

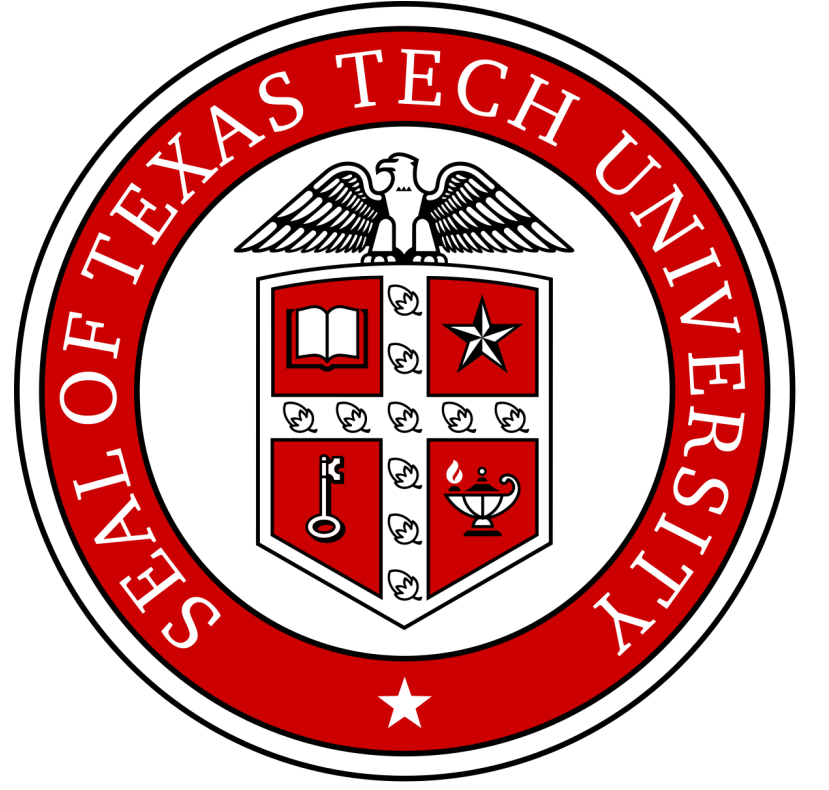


Renormalized Area of Surfaces in \mathbb{H}^3

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Historical Background

Motivated by the physical theory of the AdS/CFT correspondence of J. Maldacena, the renormalized area of minimal submanifolds was introduced in 1999 by C. Graham and E. Witten, who defined it as the Hadamard regularization of the area. The specific case of surfaces approaching the boundary at infinity at a right angle was further analyzed by S. Alexakis and R. Mazzeo in 2010, who found a connection between the renormalized area of minimal surfaces and the Willmore energy of such surfaces viewed as Euclidean surfaces.

Renormalized Area and Bending Energy

Let M be a smooth and compact surface properly embedded in the **hyperbolic space** \mathbb{H}^3 and assume that M approaches the ideal boundary $\partial_\infty \mathbb{H}^3$ at a right angle. The area of M is infinite, but it possesses a well-defined **renormalized area** \mathcal{A}_R . In 2010, S. Alexakis and R. Mazzeo obtained an explicit Gauss-Bonnet type formula

$$\mathcal{A}_R(M) = -2\pi\chi(M) - \frac{1}{2} \int_M |\hat{B}|^2 dA + \int_M H^2 dA. \quad (1)$$

Here, $\chi(M)$ is the Euler characteristic, \hat{B} is the trace-free second fundamental form, and H represents the mean curvature of M in \mathbb{H}^3 .

After a conformal change of metric, M can be regarded as a surface in the upper half-space of \mathbb{R}^3 whose boundary ∂M lies in the plane $\{z = 0\}$ and meets this plane at a right angle. The Euclidean objects of M will be denoted with an upper bar.

If the surface is **minimal** in \mathbb{H}^3 , S. Alexakis and R. Mazzeo showed a relation between its renormalized area and its Willmore energy (viewed as a surface in \mathbb{R}^3).

In the more general case, we have:

Theorem ([2, 3])

The renormalized area of a surface M in \mathbb{H}^3 approaching $\partial_\infty \mathbb{H}^3$ at a right angle is

$$\mathcal{A}_R(M) = \int_M H^2 dA - \int_M \bar{H}^2 d\bar{A}, \quad (2)$$

the difference between the hyperbolic and Euclidean bending energies.

First Proof ([2])

On the truncated surface $\underline{M} = M \cap \{z \geq \epsilon\}$, $\epsilon > 0$, we compute the Laplacian:

$$\bar{\nabla} \cdot \bar{\nabla}(\log z) = 2\bar{H} \frac{\bar{v}_3}{z} + \frac{\bar{v}_3^2}{z^2} - \frac{1}{z^2} = \left(\bar{H} + \frac{\bar{v}_3}{z}\right)^2 - \bar{H}^2 - \frac{1}{z^2} = H^2 - \bar{H}^2 - \frac{1}{z^2},$$

where \bar{v}_3 is the vertical component of the unit normal $\bar{\nu}$ and $\bar{\nabla} \cdot$ is the surface divergence. The result then follows after rearranging, integrating over \underline{M} , applying the **divergence theorem**, and letting $\epsilon \rightarrow 0^+$.

Second Proof ([3])

From the **conformal invariance** of the Willmore energy,

$$|\hat{B}|^2 g = 2(H^2 - K - 1)g = 2(\bar{H}^2 - \bar{K})\bar{g},$$

where K is the Gaussian curvature of M . Using this in (1) and employing the classical **Gauss-Bonnet theorem**, we deduce the result.

Equilibrium Surfaces

The first variation formula for the renormalized area \mathcal{A}_R was computed by S. Alexakis and R. Mazzeo in 2010, considering expansions of the graphing function.

Here we follow an essentially different approach consisting of differentiating (2) among **admissible deformations**, that is, variations tangent to the plane $\{z = 0\}$ that preserve the orthogonality of the intersection with this plane. We then have:

Proposition ([2])

Equilibrium surfaces for \mathcal{A}_R are minimal surfaces in \mathbb{H}^3 which satisfy $\partial_n \bar{H} = 0$ along ∂M . (Here n is the Euclidean outward pointing conormal to ∂M .)

Employing a refined version of the asymptotic maximum principle, S. Alexakis and R. Mazzeo showed that equilibria with **connected boundary** are totally geodesic discs of \mathbb{H}^3 .

Our following theorem, which applies to genus zero surfaces with an arbitrary number of boundary components, complements this result:

Theorem ([2])

Genus zero equilibria for the renormalized area \mathcal{A}_R are totally geodesic discs of \mathbb{H}^3 .

Proof

Let $\bar{\Phi}$ be the Euclidean quadratic Hopf differential and construct $\Phi = \bar{\Phi}/z$, the **hyperbolic Hopf differential**. From the equilibrium conditions, Φ can be holomorphically reflected to a closed doubled surface of genus $b - 1$ (for b boundary components).

From the **four-vertex theorem**, we obtain that at each boundary component the extended Φ has, at least, four zeros. This forces Φ to be identically zero. Therefore, $\bar{\Phi} \equiv 0$ and the original surface is totally umbilical in \mathbb{R}^3 .

Spherical Catenoids

Definition. A non-totally geodesic spherical rotational minimal surface M in \mathbb{H}^3 is a **spherical catenoid**.

Characterization of Generating Curves

Consider the **Blaschke's bending functional** defined by

$$\Theta(\gamma) := \int_\gamma \sqrt{\kappa} ds,$$

where κ is the curvature of the **hyperbolic** curve $\gamma : I \subseteq \mathbb{R} \rightarrow \mathbb{H}^2$ parameterized by the arc length $s \in I$. The first integral of its associated Euler-Lagrange equation is

$$\kappa_s^2 = 4\kappa^2(-\kappa^2 + 2a\kappa + 1), \quad a \in \mathbb{R}. \quad (3)$$

Definition. Curves whose curvature κ is a solution of (3) are **1/2-elastic curves**.

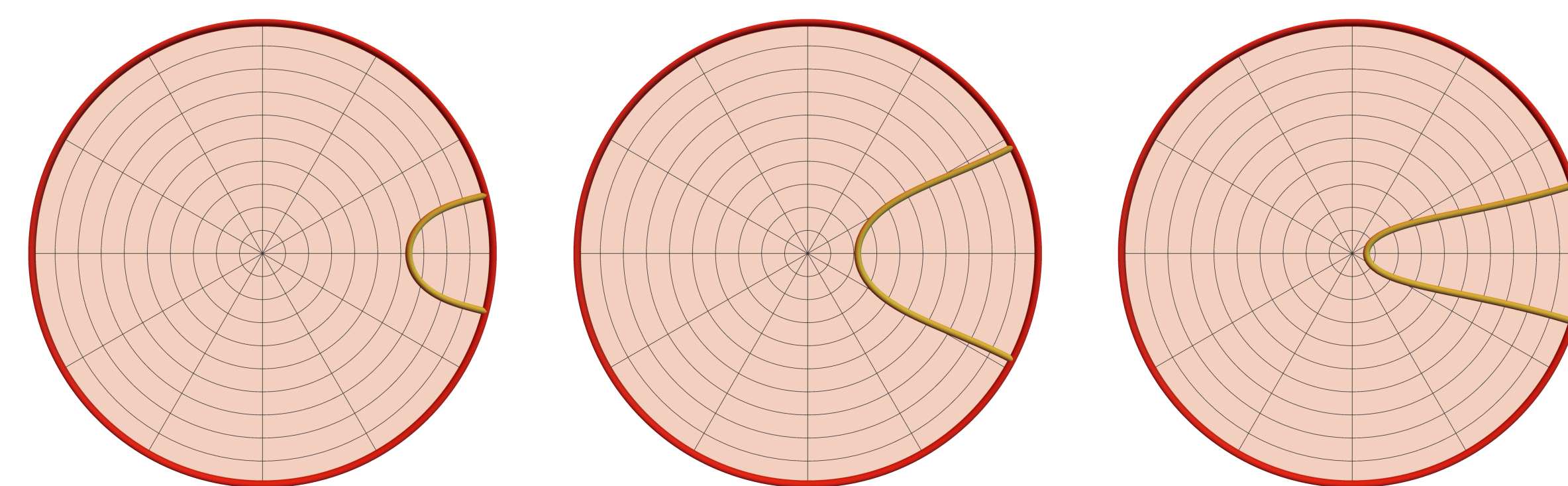


Figure. Three 1/2-elastic curves for $a > 0$ in the Poincaré disc model for \mathbb{H}^2 .

Theorem ([1, 4])

The generating curves of spherical catenoids are 1/2-elastic curves with $a > 0$.

Consequently, spherical catenoids form a one-parameter family of surfaces M_ζ in \mathbb{H}^3 , which can be indexed by the parameter

$$\zeta := \sqrt{\frac{a + \sqrt{a^2 + 1}}{2\sqrt{a^2 + 1}}} \in \left(\sqrt{2}/2, 1\right).$$

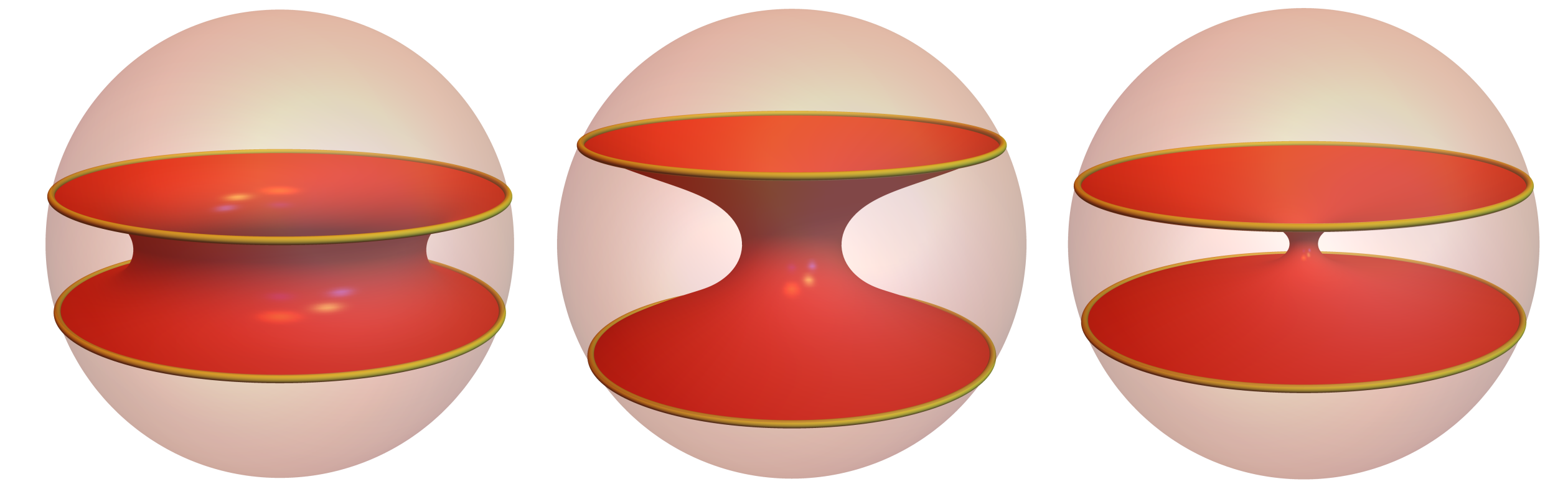


Figure. Three spherical catenoids M_ζ in the Poincaré ball model for \mathbb{H}^3 .

Renormalized Area

Combining the above characterization and the identity (1), we show:

Theorem ([4])

The renormalized area of the spherical catenoids M_ζ , $\zeta \in (\sqrt{2}/2, 1)$, in \mathbb{H}^3 is

$$\mathcal{A}_R(M_\zeta) = \frac{4\pi}{\sqrt{2\zeta^2 - 1}} \left((1 - \zeta^2)K(\zeta) - E(\zeta) \right), \quad (4)$$

where K and E are the complete elliptic integrals of the first and second kind.

As a consequence of (4), we get the following limits:

$$\lim_{\zeta \rightarrow \sqrt{2}/2^+} \mathcal{A}_R(M_\zeta) = -\infty \quad \text{and} \quad \lim_{\zeta \rightarrow 1^-} \mathcal{A}_R(M_\zeta) = 2\mathcal{A}_R(\mathbb{H}^2) = -4\pi.$$

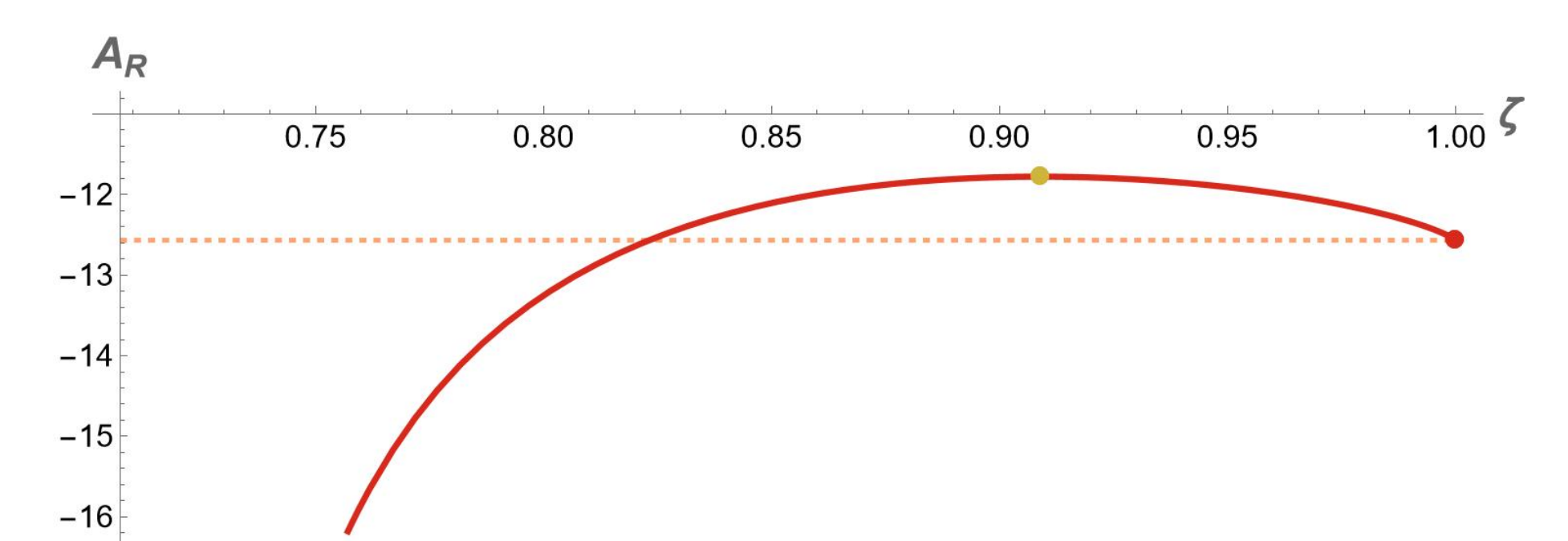


Figure. Graph of $\mathcal{A}_R(M_\zeta)$, $\zeta \in (\sqrt{2}/2, 1)$. The dashed line represents $2\mathcal{A}_R(\mathbb{H}^2) = -4\pi$.

Furthermore, the spherical catenoid M_{ζ^*} which attains the maximum of $\mathcal{A}_R(M_\zeta)$ is geometrically special:

Corollary ([4])

The maximum value of $\mathcal{A}_R(M_\zeta)$ is attained at the catenoid with maximum height.

References

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