LECTURE NOTES

Math 2360, Linear Algebra

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1 Systems of Linear Equations (Chapter 1)

Definition 1.1 *An affine equation in n variables* $x_1,...,x_n$ *is an equation of the form*

$$a_1x_1 + ... + a_nx_n = b\,.$$

The coefficients $a_1,...,a_n$ *are real numbers as well as the constant term* b*. If* $b = 0$*, the equation is said to be linear.*

Remark 1.2 *Throughout these lecture notes, we will simply refer to the above equation as linear, regardless of the value of* b*.*

Definition 1.3 *A solution of a linear equation in* n *variables is a sequence of* n *real numbers* $s_1,...,s_n$ *that satisfy the equation. The set of all solutions of a linear equation is its solution set.*

Proposition 1.4 (Number of Solutions) *The solution set of a linear equation in* n *variables depends on* $n-1$ *parameters.*

Proposition 1.5 (Geometrical Meaning) *A linear equation in* n *variables represents an hyperplane in* $\mathbb{R}^n$*. If* $n = 2$*, we have a line in the plane* $\mathbb{R}^2$*, and if* $n = 3$ *we have a plane in the space* $\mathbb{R}^3$*.*

Example 1.6 *Solve the following linear equations:*

1. $x_1 + 2x_2 = 4.$
2. $3x + 2y - z = 3.$

Definition 1.7 *A system of m linear equations in n variables is a set of m equations, each of which is a linear equation in n variables, i.e.,*

$$\begin{aligned} a_{11}x_1 + ... + a_{1n}x_n &= b_1 \,, \\ &\vdots \\ a_{m1}x_1 + ... + a_{mn}x_n &= b_m \,. \end{aligned}$$

For simplicity, the above system is called linear system. A solution of a linear system is a sequence of numbers s_1,...,s_n that is a solution of all the equations in the system.

Example 1.8 *Find the solution to the linear system:*

$$\begin{cases} 3x_1 + 2x_2 = 1 \\ -x_1 + x_2 = 4 \end{cases} .$$

(Answer: $x_1 = -7/5$ and $x_2 = 13/5$.)

Proposition 1.9 (Geometric Meaning) *The solution of a linear system (with m linear equations in n variables) represents the intersection of m hyperplanes in $\mathbb{R}^n$.*

Proposition 1.10 (Number of Solutions) *For a linear system, one of the following assertions holds:*

1. *The system has exactly one solution (we say that the system is consistent or compatible).*
2. *The system has infinitely many solutions (we say that the system is consistent or compatible).*
3. *The system has no solution (we say that the system is inconsistent or incompatible).*

Example 1.11 *Obtain the solutions, if possible, for the following systems:*

1. *Two intersecting lines:*

$$\begin{cases} x + y = 3 \\ x - y = -1 \end{cases} .$$

(Answer: $x = 1$ and $y = 2$.)

2. *Two coincident lines:*

$$\begin{cases} x + y = 3 \\ 2x + 2y = 6 \end{cases} .$$

3. *Two parallel lines:*

$$\begin{cases} x + y = 3 \\ x + y = 1 \end{cases} .$$

Remark 1.12 *If the system has several equations and several variables, solving the system by substitution may be tedious.*

Proposition 1.13 (Gaussian Elimination) *To solve a system of linear equations one can apply the following steps, without modifying the solution:*

1. *Interchange two equations.*
2. *Multiply an equation by a nonzero constant.*
3. *Add an equation to another equation.*

Example 1.14 *Solve the following systems of linear equations using the Gaussian elimination process:*

1. *A compatible system with one solution (three planes intersecting in a point):*

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases} .$$

   (Answer: $x = 1$, $y = -1$ and $z = 2$.)

2. *An incompatible system:*

$$\begin{cases} x_1 - 3x_2 + x_3 = 1 \\ 2x_1 - x_2 - 2x_3 = 2 \\ x_1 + 2x_2 - 3x_3 = -1 \end{cases} .$$

3. *A compatible system with infinitely many solutions (three planes intersecting in a line):*

$$\begin{cases} x_2 - x_3 = 0 \\ x_1 - x_3 = -1 \\ -x_1 + 3x_2 - 2x_3 = 1 \end{cases} .$$

Remark 1.15 *It is convenient to introduce the notation of matrix in order to apply the Gaussian elimination in a more simplified way.*

Definition 1.16 *If m and n are natural numbers, an $m \times n$ matrix is a rectangular array*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

in which each entry a_{ij} is a number. An $m \times n$ matrix has m rows and n columns. If $m = n$, the matrix is a square matrix.

Remark 1.17 *If we have a system of linear equations, we can construct the coefficient matrix by introducing the matrix whose entries are the corresponding coefficients. Similarly, we can construct the augmented matrix by including one column more to the right with the numbers* b_i, $i = 1, ..., m$. *We then apply the Gaussian elimination steps to this matrix, in order to solve the system.*

Example 1.18 *Solve the following systems of linear equations:*

1. *Compatible system with one solution:*

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}.$$

(Answer: $x = 1$, $y = -1$ *and* $z = 2$.*)*

2. *Compatible system with one solution:*

$$\begin{cases} x_2 + x_3 - 2x_4 = -3 \\ x_1 + 2x_2 - x_3 = 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 = -2 \\ x_1 - 4x_2 - 7x_3 - x_4 = -19 \end{cases}.$$

(Answer: $x_1 = -1$, $x_2 = 2$, $x_3 = 1$ *and* $x_4 = 3$.*)*

3. *A system with no solution:*

$$\begin{cases} x_1 - x_2 + 2x_3 = 4 \\ x_1 + x_3 = 6 \\ 2x_1 - 3x_2 + 5x_3 = 4 \\ 3x_1 + 2x_2 - x_3 = 1 \end{cases}.$$

4. *A system with infinitely many solutions:*

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 0 \\ 3x_1 + 5x_2 = 1 \end{cases}.$$

Definition 1.19 *Systems of linear equations in which each of the constant term is zero are called homogeneous.*

Proposition 1.20 *Every homogeneous system of linear equations is consistent and so, it has at least one solution (the trivial solution, i.e.,* $x_1 = ... = x_n = 0$*). Moreover, if the system has fewer equations than variables, then it must have infinitely many solutions.*

1.1 Exercises

1. Solve the system of linear equations:

$$\begin{cases} x_1 - 3x_3 = -2 \\ 3x_1 + x_2 - 2x_3 = 5 \\ 2x_1 + 2x_2 + x_3 = 4 \end{cases} .$$

   (Answer: $x_1 = 4$, $x_2 = -3$ and $x_3 = 2$.)

2. Solve the system of linear equations:

$$\begin{cases} 2x + y - z + 2t = -6 \\ 3x + 4y + t = 1 \\ x + 5y + 2z + 6t = -3 \\ 5x + 2y - z - t = 3 \end{cases} .$$

   (Answer: $x = 1$, $y = 0$, $z = 4$ and $t = -2$.)

3. * Show that two planes (affine subspaces of dimension two) in $\mathbb{R}^4$ can intersect in a point.

4. * Show that two planes in $\mathbb{R}^4$ can be skew (i.e., they do not intersect and they are not parallel).

2 Matrices (Chapter 2)

Definition 2.1 *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices of size $m \times n$. Then their sum is the $m \times n$ matrix*

$$A + B = (a_{ij} + b_{ij}) .$$

Remark 2.2 *The sum of matrices is only defined for matrices of the same size.*

Definition 2.3 *Let $A = (a_{ij})$ be an $m \times n$ matrix and consider $c \in \mathbb{R}$. Then the scalar multiple of A by c is the $m \times n$ matrix $cA = (ca_{ij})$.*

Example 2.4 *Consider the matrices:*

$$A = \begin{pmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix} , \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{pmatrix} .$$

Compute $3A$, $-B$ and $3A - B$.

Definition 2.5 *Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ an $n \times p$ matrix. Then the product of A and B is the matrix of size $m \times p$ given by*

$$AB = A \cdot B = (c_{ij}) ,$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

Remark 2.6 *Note that the product BA may not be defined. Indeed, it is only defined when the matrices are square matrices. Moreover, even in this case, $AB \neq BA$, in general.*

Example 2.7 *Compute the product of the following matrices:*

$$A = \begin{pmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{pmatrix} , \qquad \text{and} \qquad B = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} .$$

Answer:

$$AB = \begin{pmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{pmatrix} .$$

Definition 2.8 *The matrix*

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

is the identity matrix of order n.

Proposition 2.9 *The matrix I_n serves as the identity for matrix multiplication. That is, for any matrix A of size $m \times n$, $I_m A = A I_n = A$ holds.*

Definition 2.10 *The transpose of a matrix A of size $m \times n$, is the matrix A^T of size $n \times m$ formed by writing the rows of A as columns.*

Example 2.11 *Find the transpose of the following matrix:*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

Proposition 2.12 (Properties of the Transpose) *Let A and B be two matrices of suitable sizes. Then, the following properties hold:*

1. *The transpose of a transpose is the original matrix, i.e., $\left(A^T\right)^T = A$.*
2. *The transpose of the sum is the sum of the transposes, i.e., $(A+B)^T = A^T + B^T$.*
3. *The transpose of the product is the product of the transposes, multiplied on the other side, i.e., $(AB)^T = B^T A^T$.*

Example 2.13 *Show the last property for the following matrices:*

$$A = \begin{pmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{pmatrix}, \qquad \text{and} \qquad B = \begin{pmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{pmatrix}.$$

Definition 2.14 *A square matrix is symmetric if it is equal to its transpose. In other words if the rows coincide with the columns.*

Proposition 2.15 *The product of a matrix A with its transpose A^T is a symmetric matrix.*

Definition 2.16 *A square matrix A is invertible (or, also, nonsingular) when there exists a square matrix B of the same size such that*

$$AB = I_n = BA\,.$$

The matrix B is the inverse of A (which is unique) and it is usually denoted by A^{-1}.

Proposition 2.17 *The inverse of a matrix is unique.*

Remark 2.18 *There are three methods to obtain the inverse of a matrix:*

1. *Solving a system of linear equations, i.e.,* $AB = I$*.*

2. *Gauss-Jordan elimination, i.e., considering the matrix* $(A \,|\, I)$ *and applying row transformations until we obtain* $(I \,|\, A^{-1})$*.*

3. *Applying this formula:*

$$A^{-1} = \frac{1}{|A|}\mathrm{Adj}(A)\,,$$

where $|A|$ *denotes the determinant of* A *and* $\mathrm{Adj}(A)$ *is the adjoint. (We need Chapter 3).*

Example 2.19 *If possible, compute the inverse of the following matrices using the first two methods:*

1. *Nonsingular matrix:*

$$A = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}.$$

Answer:

$$A^{-1} = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}.$$

2. *Nonsingular matrix:*

$$B = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix}.$$

Answer:

$$B^{-1} = \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix}.$$

3. *Singular matrix:*

$$C = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{pmatrix}.$$

Proposition 2.20 (Properties of Inverse Matrices) *Let* A *and* B *be two invertible square matrices of the same size. Then, the following properties hold:*

1. *The inverse of the inverse is the original matrix, i.e.,* $\left(A^{-1}\right)^{-1} = A$*.*

2. *The inverse of the product is the product of the inverses, multiplied on the other side, i.e.,* $(AB)^{-1} = B^{-1}A^{-1}$*.*

Theorem 2.21 *If* A *is an invertible matrix, then the system of linear equations* $AX = B$ *has a unique solution* $X = A^{-1}B$*. Here,* X *is the vector (*$m \times 1$ *matrix) having the unknowns and* B *is the vector with the constants* b*.*

Example 2.22 *Use the inverse matrix to solve the following system of linear equations:*

$$\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1 \\ 2x + 4y + z = -2 \end{cases}.$$

(Answer: $x = 2$, $y = -1$ *and* $z = -2$.*)*

2.1 Exercises

1. * Prove Proposition 2.15.

2. Solve the system of linear equations using all methods already studied:

$$\begin{cases} 2x + 3y + z = 4 \\ 3x + 3y + z = 8 \\ 2x + 4y + z = 5 \end{cases}.$$

   (Answer: $x = 4$, $y = 1$ and $z = -7$.)

3. Solve Exercises 1 and 2 of previous section using the inverse matrix.

3 Determinants (Chapter 3)

Remark 3.1 *The determinant is a multilinear form that associates a number to each square matrix. In particular, if the square matrix is of size* 1×1*, then the determinant is the only entry of a matrix.*

Definition 3.2 *Let* $A = (a_{ij})$ *be a square matrix. The* minor M_{ij} *of the entry* a_{ij} *is the determinant of the matrix obtained by deleting the* i*-th row and the* j*-th column of* A*. The* cofactor C_{ij} *of the entry* a_{ij} *is defined by* $C_{ij} = (-1)^{i+j} M_{ij}$*.*

Definition 3.3 *Let* $A = (a_{ij})$ *be a square matrix of order* $n \geq 2$*, then the* determinant *of* A *is the sum of the entries in the first row of* A *multiplied by their respective cofactors. That is,*

$$\det(A) = |A| = \sum_{j=1}^{n} a_{1j} C_{1j} \,.$$

Proposition 3.4 (Determinant of a 2×2 Matrix) *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then, the determinant of A *is given by* $|A| = a_{11}a_{22} - a_{12}a_{21}$*.*

Example 3.5 *Compute the determinants of the following matrices:*

1. *Determinant of a matrix of order* 1*:*

$$A = (3) \,.$$

   (Answer: 3*.)*

2. *Determinant of a matrix of order* 2*:*

$$B = \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}.$$

   (Answer: 7*.)*

3. *Determinant of a matrix of order* 3*:*

$$C = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix}.$$

   (Answer: 14*.)*

4. *Determinant of a matrix of order* 4*:*

$$D = \begin{pmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{pmatrix}.$$

(Answer: 39*.)*

Proposition 3.6 *Let $A = (a_{ij})$ be a square matrix of order n. Then the determinant of A can be computed as*

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij} C_{ij} ,$$

for any $i = 1, ..., n$ (or also for any $j = 1, ..., n$ changing the summation index).

Example 3.7 *Compute the determinant of the matrix D in previous example expanding by the row or column of your choice.*

Remark 3.8 *For a square matrix of order* 3*, we can apply the rule of Sarrus. Consider the general matrix of order* 3

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then, the determinant of A is given by

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} .$$

Proposition 3.9 (Determinant of a Triangular Matrix) *If A is a triangular matrix (i.e., it has all zeros over or below the diagonal) of order n, then the determinant is the product of the entries on the diagonal. That is,*

$$|A| = a_{11}...a_{nn} .$$

Proposition 3.10 *Let A and B be square matrices of order n. Then the following assertions hold:*

1. *When B is obtained from A by interchanging two rows of A, then $|B| = -|A|$.*
2. *When B is obtained from A by multiplying a row of A by a constant $c \in \mathbb{R}$, then $|B| = c|A|$.*
3. *When B is obtained from A by adding a multiple of a row of A to another row of A, then $|B| = |A|$.*
4. *The same properties are true for columns.*

5. *If $B = cA$ for some constant $c \in \mathbb{R}$, then $|B| = c^n|A|$.*

Proposition 3.11 *If A is a square matrix, then $|A| = |A^T|$.*

Example 3.12 *Compute the determinant of the following matrix:*

$$A = \begin{pmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{pmatrix}.$$

(Answer: -135.)

Theorem 3.13 *Let A be a square matrix. The matrix A is invertible (nonsingular) if and only if $|A| \neq 0$.*

Proposition 3.14 *If A is an invertible matrix, then $|A| = 1/|A^{-1}|$.*

Definition 3.15 *Let $A = (a_{ij})$ be a square matrix. The transpose of the matrix whose entries are the cofactors of A is called the adjoint (or, adjugate) of A. If C_{ij} denotes the cofactor of the entry a_{ij}, then the adjoint of A is*

$$\mathrm{Adj}(A) = \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & Cnn \end{pmatrix}.$$

Theorem 3.16 *If A is an invertible matrix, then*

$$A^{-1} = \frac{1}{|A|}\mathrm{Adj}(A)\,.$$

Example 3.17 *Use the adjoint to compute the inverse of*

$$A = \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{pmatrix}.$$

Answer:

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{pmatrix}.$$

Theorem 3.18 (Cramer's Rule) *Let A be the coefficient matrix of a system of n linear equations in n variables and assume that A is invertible. Then the solution of the system is given by*

$$x_1 = \frac{|A_1|}{|A|}, \qquad \dots \qquad x_n = \frac{|A_n|}{|A|},$$

where, for any $i = 1, \dots, n$, A_i denotes the matrix composed by the columns of A once we substitute the i-th column by the constants in the system of linear equations.

Example 3.19 *Use Cramer's Rule to solve the system of linear equations:*

$$\begin{cases} -x + 2y - 3z = 1 \\ 2x + z = 0 \\ 3x - 4y + 4z = 2 \end{cases}.$$

(Answer: $x = 4/5$, $y = -3/2$ and $z = -8/5$.)

Remark 3.20 *At this point, we can obtain all the real solutions, when they exist, of systems of linear equations. However, consider the following simple problem: "A store sells boxes of donuts at \$7 each and pizzas at \$18 each. If in one day they have received \$208, how many pizzas and boxes of donuts were sold?"*

See Appendix A for more information about systems of linear equations with integer solutions.

3.1 Exercises

1. Compute the determinant of the matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

   (Answer: -3.)

2. Compute the inverse of the matrix:

$$A = \begin{pmatrix} 4 & -1 & 1 \\ 2 & 2 & 3 \\ 5 & -2 & 6 \end{pmatrix}$$

   using the adjoint. Answer:

$$A^{-1} = \frac{1}{55} \begin{pmatrix} 18 & 4 & -5 \\ 3 & 19 & -10 \\ -14 & 3 & 10 \end{pmatrix}.$$

3. Solve the system of linear equations

$$\begin{cases} 4x_1 - x_2 + x_3 = -5 \\ 2x_1 + 2x_2 + 3x_3 = 10 \\ 5x_1 - 2x_2 + 6x_3 = 1 \end{cases}$$

   using Cramer's Rule. (Answer: $x_1 = -1$, $x_2 = 3$ and $x_3 = 2$.)

4. Solve the systems of linear equations of Section 1 using Cramer's Rule.

5. * Write the determinant of the general matrix of order 4:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

6. * Compute the determinant of the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}.$$

Review Problems

1. Solve the following systems of linear equations using all methods:

   (a)

$$\begin{cases} x + 2y - z + 3t = -2 \\ -x + y - 2t = 0 \\ 3x - y + 2z - t = 9 \\ -2x - 2y + 3z - t = 0 \end{cases} .$$

   (Answer: $x = 2$, $y = 0$, $z = 1$ and $t = -1$.)

   (b)

$$\begin{cases} x + y + z + t = 0 \\ 2x - y - z + 2t = 0 \\ -x + y - z + 2t = -1 \\ x + y - z - 2t = 5 \end{cases} .$$

   (Answer: $x = 1$, $y = 1$, $z = -1$ and $t = -1$.)

   (c)

$$\begin{cases} x + 2y - z = -1 \\ -x - y - 2z = 0 \\ 2x + y + z = 1 \end{cases} .$$

   (Answer: $x = 1$, $y = -1$ and $z = 0$.)

   (d)

$$\begin{cases} x + y + z = 1 \\ -x + y - z = -1 \\ x - y - z = -3 \end{cases} .$$

   (Answer: $x = -1$, $y = 0$ and $z = 2$.)

4 Vector Spaces (Chapter 4)

Remark 4.1 *Roughly speaking, a vector v is a mathematical object which has both:*

(i) Magnitude (norm), and

(ii) Direction.

It then can be represented by an arrow. A set of vectors is a vector space. However, the precise mathematical definition requires some axioms and previous constructions.

Definition 4.2 (Group) *A group is a set G together with a closed binary operation, denoted for convenience by $+$, (i.e., for any $a, b \in G$, $a + b \in G$) satisfying:*

1. *The operation is associative. For all $a, b, c \in G$,*

$$(a + b) + c = a + (b + c) .$$

2. *Existence of the identity element. There exists an element $0 \in G$ (the notation is again for convenience) such that for every $a \in G$,*

$$a + 0 = a = 0 + a .$$

3. *Existence of inverse elements. For every $a \in G$, there exists an element $b \in G$ such that*

$$a + b = 0 = b + a .$$

Proposition 4.3 *The inverse of $a \in G$ is unique. For convenience, we denote it by $-a$.*

Example 4.4 *The followings are examples (or not) of groups:*

1. *The natural numbers with the usual addition $(\mathbb{N}, +)$ is not a group.*
2. *The integer numbers with the usual addition $(\mathbb{Z}, +)$ is a group.*
3. *The integer numbers with the usual multiplication $(\mathbb{Z}, \cdot)$ is not a group.*
4. *The real numbers with the usual multiplication $(\mathbb{R}, \cdot)$ is not a group.*
5. *The nonzero real numbers with the usual multiplication $(\mathbb{R} \setminus \{0\}, \cdot)$ is a group.*
6. *The set of all invertible real matrices with the multiplication $\mathrm{GL}(n, \mathbb{R})$ is a group, called the general linear group.*

Definition 4.5 (Abelian Group) *A group $(G, +)$ is said to be abelian (or, commutative) if the operation $+$ is commutative, i.e., for all $a, b \in G$,*

$$a + b = b + a .$$

Remark 4.6 *All groups are not abelian. For instance,* $\mathrm{GL}(n, \mathbb{R})$ *is not abelian because the multiplication of matrices is not commutative.*

Remark 4.7 *See the Appendix B for extra information about Crystallographic Groups and Tessellations.*

Definition 4.8 *A field is a set* F *together with two closed binary operations, addition and multiplication denoted by* $+$ *and* $\times$*, respectively, and such that the following hold:*

1. $(F, +)$ *is an abelian group.*
2. $(F \setminus \{0\}, \times)$ *is an abelian group.*
3. *Distributivity of multiplication over addition, i.e.,*

$$a \times (b + c) = (a \times b) + (a \times c)\,.$$

Example 4.9 *The followings are examples of fields:*

1. *Rational number* $(\mathbb{Q}, +, \times)$.
2. *Real numbers* $(\mathbb{R}, +, \times)$.
3. *Complex numbers* $(\mathbb{C}, +, \times)$.
4. *Fields of characteristic prime* $(\mathbb{Z}_p, +, \times)$.

Remark 4.10 *Throughout this course, when we speak about a field we will understand that this field is* $(\mathbb{R}, +, \times)$.

Definition 4.11 *A vector space over a field* F *is an abelian group* $(V, +)$ *together with a closed scalar multiplication, denoted by* $\cdot$*, satisfying:*

1. *The scalar multiplication is associative. For any* $a, b \in F$ *and* $v \in V$*,*

$$a \cdot (b \cdot v) = (ab) \cdot v\,,$$

   where ab *is the multiplication on the field* F.

2. *Existence of the identity element. There exists* $1 \in F$ *(the multiplicative identity in* F*) such that* $1v = v$*, for all* $v \in V$.
3. *Distributivity of scalar multiplication with respect to vector addition. For any* $u, v \in V$ *and* $a \in F$*,*

$$a \cdot (u + v) = a \cdot u + a \cdot v\,.$$

4. *Distributivity of scalar multiplication with respect to field addition. For any $a, b \in F$ and $v \in V$,*

$$(a + b) \cdot v = a \cdot v + b \cdot v\,.$$

Remark 4.12 *From now on, the scalar multiplication will be denoted by av, without $\cdot$.*

Example 4.13 *The followings are examples of vector spaces:*

1. *The space $\mathbb{R}^n$ with the standard operations.*
2. *The vector space of polynomials of degree n or less, $P_n(x)$.*
3. *The vector space of $m \times n$ matrices.*
4. *The vector space of continuous functions defined on a fixed interval.*

4.1 Subspaces of Vector Spaces

Definition 4.14 *A nonempty subset W of a vector space V is a subspace (or, vector subspace, or linear subspace) of V when W is a vector space under the operations defined in V.*

Proposition 4.15 *Let W be a nonempty subset of V. The subset W is a subspace of V if and only if:*

1. *If u and v are in W, then $u + v$ is in W.*
2. *If u is in W and c is any scalar, then cu is in W.*

Example 4.16 *The followings are examples of subsets (or not):*

1. *The zero subspace is a subspace of any vector space V.*
2. *The vector space V is a subspace of V.*
3. *The subset $\{(x, y) \in \mathbb{R}^2 \,|\, x + y = 0\}$ is a subspace of $\mathbb{R}^2$.*
4. *The subset $\{(x, y) \in \mathbb{R}^2 \,|\, x + y = 1\}$ is not a subspace of $\mathbb{R}^2$.*
5. *The subset $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ is not a subspace of $\mathbb{R}^2$.*
6. *The set of symmetric matrices of order n is a subspace of the vector space of square matrices of order n.*
7. *The set of singular matrices is not a subspace.*

Proposition 4.17 *Let U and W be two subspaces of a vector space V. Then the intersection of U and W, $U \cap W$, is a subspace of V.*

Example 4.18 *Describe all the subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$.*

4.2 Basis and Dimension

Definition 4.19 *A vector v in a vector space V is a* <u>*linear combination*</u> *of the vectors u_1,...,u_k, in V if*

$$v = c_1 u_1 + \ldots + c_k u_k \,,$$

for some scalars c_1,...,c_k.

Example 4.20 *The vector $v_1 = (1,3,1)$ is a linear combination of $v_2 = (0,1,2)$ and $v_3 = (1,0,-5)$. (Answer: $v_1 = 3v_2 + v_3$.)*

Definition 4.21 *Let $\mathcal{S} = \{v_1, \ldots, v_k\}$ be a subset of a vector space V. The set $\mathcal{S}$ is a* <u>*spanning set*</u> *of V when every vector in V can be written as a linear combination of vectors in $\mathcal{S}$. In such cases, it is said that $\mathcal{S}$* <u>*spans*</u> *V.*

Example 4.22 *The following sets span $\mathbb{R}^3$:*

1. $\mathcal{S} = \{(1,0,0),(0,1,0),(1,2,1),(0,0,1)\}$.
2. $\mathcal{S} = \{(1,2,3),(0,1,2),(-2,0,1)\}$.

Definition 4.23 *A set of vectors $\mathcal{S} = \{v_1, \ldots, v_k\}$ in a vector space V is* <u>*linearly independent*</u> *when the vector equation*

$$c_1 v_1 + \ldots + c_k v_k = 0 \,,$$

has only the trivial solution $c_1 = \ldots = c_k = 0$. If there are also nontrivial solutions, then $\mathcal{S}$ is <u>*linearly dependent*</u>*.*

Example 4.24 *Determine if the set $\mathcal{S} = \{(1,2,3),(0,1,2),(-2,0,1)\}$ is linearly independent.*

Definition 4.25 *A set of vectors $\mathcal{B} = \{v_1, \ldots, v_n\}$ in a vector space V is a* <u>*basis*</u> *for V when:*

(i) $\mathcal{B}$ spans V, and

(ii) $\mathcal{B}$ is linearly independent.

Remark 4.26 *There exist vector spaces whose bases have infinitely many vectors. Those vector spaces are called* <u>*infinite dimensional*</u>*. We are going to restrict ourselves to* <u>*finite dimensional*</u> *vector spaces.*

Theorem 4.27 *If a vector space V has one basis with n vectors, then every bases for V has exactly n vectors.*

Definition 4.28 *If a vector space V has a basis consisting of n vectors, then the number n is the* <u>*dimension*</u> *of V. We denote it by $\dim(V) = n$.*

Example 4.29 *Prove that the followings are bases for $\mathbb{R}^3$ and so $\dim(\mathbb{R}^3) = 3$:*

1. $\mathcal{B}_s = \{(1,0,0),(0,1,0),(0,0,1)\}$ *(<u>standard</u> or <u>canonical</u> basis).*
2. $\mathcal{B} = \{(1,2,3),(0,1,2),(-2,0,1)\}$.

4.3 Coordinates and Change of Basis

Definition 4.30 *Let $\mathcal{B} = \{v_1, ..., v_n\}$ be a basis for a vector space V and let x be a vector in V such that*

$$x = c_1 v_1 + ... + c_n v_n \,.$$

The scalars c_1,...,c_n are the <u>*coordinates*</u> *of x with respect to the basis $\mathcal{B}$. The* <u>*coordinate matrix*</u> *of x relative to $\mathcal{B}$ is the column matrix:*

$$X_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} .$$

Remark 4.31 *It is also possible to write everything by rows, instead of by columns. In this course, we will follow the notation by columns.*

Example 4.32 *Obtain the coordinates of the vector $x = (-2, 1, 3)$ with respect to the basis:*

1. $\mathcal{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. *Answer:*

$$X_{\mathcal{B}} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} .$$

2. $\mathcal{B}' = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$. *Answer:*

$$X_{\mathcal{B}'} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} .$$

Definition 4.33 *Let $\mathcal{B} = \{v_1, ..., v_n\}$ and $\mathcal{B}' = \{u_1, ..., u_n\}$ be two bases for a vector space V. Assume*

$$\begin{aligned} v_1 &= c_{11}u_1 + ... + c_{n1}u_n \,, \\ &\vdots \\ v_n &= c_{1n}u_1 + ... + c_{nn}u_n \,. \end{aligned}$$

Then, the <u>*transition matrix*</u> *from $\mathcal{B}$ to $\mathcal{B}'$ is*

$$P = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \dots & c_{nn} \end{pmatrix} .$$

Proposition 4.34 *Assume P is the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$, then P is invertible and the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$ is, precisely, P^{-1}.*

Example 4.35 *Compute:*

1. *The transition matrix from* $\mathcal{B}_s = \{(1,0,0),(0,1,0),(0,0,1)\}$ *to* $\mathcal{B}' = \{(1,2,3),(0,1,2),(-2,0,1)\}$.
   Answer:
$$P = \begin{pmatrix} -1 & 4 & -2 \\ 2 & -7 & 4 \\ -1 & 2 & -1 \end{pmatrix}.$$

2. *The transition matrix from* $\mathcal{B}' = \{(1,2,3),(0,1,2),(-2,0,1)\}$ *to* $\mathcal{B}_s = \{(1,0,0),(0,1,0),(0,0,1)\}$.
   Answer:
$$P^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}.$$

3. *Check that one is the inverse of the other one.*

Proposition 4.36 (Change of Coordinates) *Let* P *be the transition matrix from* $\mathcal{B}$ *to* $\mathcal{B}'$*. Denote by* $X_{\mathcal{B}}$ *and* $X_{\mathcal{B}'}$ *the coordinate matrices of a vector* x *relative to the bases* $\mathcal{B}$ *and* $\mathcal{B}'$*, respectively. Then,*
$$X_{\mathcal{B}'} = P X_{\mathcal{B}}\,.$$

Example 4.37 *Check the change of coordinates for the vector* $x = (-2,1,3)$ *with respect to the bases* $\mathcal{B}_s = \{(1,0,0),(0,1,0),(0,0,1)\}$ *and* $\mathcal{B}' = \{(1,2,3),(0,1,2),(-2,0,1)\}$.

4.4 Exercises

1. * Try to give an example of a group which is not clearly/directly related to Mathematics.
2. Find a basis for $\mathbb{R}^n$ and determine the dimension.
3. Show that $\mathcal{S} = \{1, x, x^2, x^3\}$ is a basis for the vector space P_3, i.e., the set of polynomials of degree less or equal three.
4. Find the coordinates of the vector $x = (1,-1,2)$ with respect to the bases:
   (a) $\mathcal{B} = \{(1,1,1),(1,1,0),(0,1,-1)\}$.
   (b) $\mathcal{B}' = \{(1,2,3),(1,0,1),(0,-1,0)\}$.

   Check the change of coordinates by computing the transition matrix.
   Answer:
$$X_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \quad X_{\mathcal{B}'} = \begin{pmatrix} 1/2 \\ 1/2 \\ 2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & -1/2 & -1/2 \\ 1 & 3/2 & 1/2 \\ -1 & -2 & -2 \end{pmatrix}.$$

5 Inner Product Spaces (Chapter 5)

Definition 5.1 *Let u, v and w be three vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $\langle u, v\rangle$ to each pair of vectors u and v which satisfies:*

(i) $\langle u, v\rangle = \langle v, u\rangle$,

(ii) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$,

(iii) $c\langle u, v\rangle = \langle cu, v\rangle$, *and*

(iv) $\langle v, v\rangle \geq 0$ *with equality if and only if* $v = 0$.

A vector space V endowed with an inner product is called an inner product space.

Remark 5.2 *An inner product on a (real) vector space V is a bilinear form (conditions (ii) and (iii)) which is both positive definite (condition (iv)) and symmetric (condition (i)). Bilinear forms can be represented by square matrices. Moreover, the symmetric condition translates to the matrix being symmetric, while so does the positive definiteness of the form.*

Example 5.3 *The followings are examples of inner product spaces:*

1. *The Euclidean space $\mathbb{R}^n$ together with the standard inner/dot/scalar product. The associated matrix is the identity.*
2. $\mathbb{R}^2$ *endowed with* $\langle u, v\rangle = u_1v_1 + 2u_2v_2$. *The associated matrix is*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Theorem 5.4 (Properties) *Let u, v and w be three vectors in an inner product space $(V, \langle\ ,\ \rangle)$, and assume c is any real number. Then the following hold:*

(i) $\langle 0, v\rangle = \langle v, 0\rangle = 0$.

(ii) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$.

(iii) $\langle u, cv\rangle = c\langle u, v\rangle$.

Definition 5.5 *Let u and v be vectors in an inner product space V.*

(i) The length (or norm) of u is $\|u\| = \sqrt{\langle u, u\rangle}$.

(ii) A vector with length one is said to be a unit vector.

(iii) *The* *distance* *between* u *and* v *is* $\mathrm{d}(u,v) = \|u - v\|$.

(iv) *The* *angle* *between two nonzero vectors is*

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|},$$

for $\theta \in [0, \pi]$.

(v) *Two vectors are said to be* *orthogonal* *if* $\theta = \pi/2$.

Proposition 5.6 *Two vectors* u *and* v *are orthogonal if and only if* $\langle u, v\rangle = 0$.

Remark 5.7 *Given a nonzero vector* v *in an inner product space* V*, the vector*

$$u = \frac{v}{\|v\|}$$

is a unit vector in the direction of v.

Remark 5.8 *For the definition of the angle we are assuming that*

$$-1 \leq \frac{\langle u, v\rangle}{\|u\|\|v\|} \leq 1,$$

which follows from Cauchy-Schwarz inequality (given in the next theorem).

Theorem 5.9 *Let* u *and* v *be vectors in an inner product space* V*. Then, the followings hold:*

(i) *Cauchy-Schwarz Inequality.* $|\langle u, v\rangle| \leq \|u\|\|v\|$.

(ii) *Triangle Inequality.* $\|u + v\| \leq \|u\| + \|v\|$.

(iii) *Pythagorean Theorem.* *The vectors* u *and* v *are orthogonal if and only if*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Theorem 5.10 (Parallelogram Law) *Let* V *be an inner product space. Then,*

$$2\|u\|^2 + 2\|v\|^2 = \|u + v\|^2 + \|u - v\|^2,$$

holds for all u *and* v *in* V.

Remark 5.11 *The parallelogram law is satisfied for norms arising from an inner product. Indeed, any norm satisfying this law comes from the inner product given by the polarization identity, which in the real case is*

$$\langle u, v\rangle = \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2\right).$$

5.1 Gram-Schmidt Process

Definition 5.12 *A set $\mathcal{S}$ of vectors in an inner product space V is* <u>*orthogonal*</u> *when every pair of vectors in $\mathcal{S}$ are orthogonal. If, in addition, each vector is a unit vector, then $\mathcal{S}$ is* <u>*orthonormal*</u>*.*

Definition 5.13 *If $\mathcal{S}$ is a basis, we will say that it is an* <u>*orthogonal basis*</u> *or, respectively,* <u>*orthonormal basis*</u>*.*

Example 5.14 *The standard basis of $\mathbb{R}^3$, $\mathcal{B}_s = \{(1,0,0),(0,1,0),(0,0,1)\}$, is an orthonormal basis with respect to the standard inner product.*

Theorem 5.15 *If $\mathcal{S} = \{v_1, ..., v_k\}$ is an orthogonal set of nonzero vectors in an inner product space V, then $\mathcal{S}$ is linearly independent.*

Corollary 5.16 *If V is an inner product space of dimension n, then any orthogonal set of n nonzero vectors is a basis for V.*

Proposition 5.17 *If $\mathcal{B} = \{v_1, ..., v_n\}$ is an orthonormal basis for an inner product space V, then the coordinate representation of a vector u relative to $\mathcal{B}$ is*

$$u = \langle u, v_1\rangle v_1 + ... + \langle u, v_n\rangle v_n \,.$$

Example 5.18 *Find the coordinate matrix of the vector $x = (5,-5,2)$ relative to the basis $\mathcal{B} = \{(3/5, 4/5, 0), (-4/5, 3/5, 0), (0,0,1)\}$. (Hint: Check first that $\mathcal{B}$ is orthonormal.)*
Answer:

$$X_{\mathcal{B}} = \begin{pmatrix} -1 \\ -7 \\ 2 \end{pmatrix}.$$

Theorem 5.19 (Gram-Schmidt) *Let $\mathcal{B} = \{v_1, ..., v_n\}$ be a basis for an inner product space V. Construct the vectors:*

$$\begin{aligned} u_1 &= v_1 \,, \\ u_2 &= v_2 - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2} u_1 \,, \\ &\vdots \\ u_n &= v_n - \frac{\langle v_n, u_1\rangle}{\|u_1\|^2} u_1 - ... - \frac{\langle v_n, u_{n-1}\rangle}{\|u_{n-1}\|^2} u_{n-1} \,. \end{aligned}$$

Then, $\mathcal{B}' = \{u_1, ..., u_n\}$ constructed as above is an orthogonal basis for V. Moreover, by making each vector a unit vector, we end up with an orthonormal basis.

Example 5.20 *Apply the Gram-Schmidt orthonormalization process to the following bases:*

1. $\mathcal{B} = \{(1,1),(0,1)\}$. *(Answer: $\mathcal{B}' = \{(1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2})\}$.)*
2. $\mathcal{B} = \{(1,1,0),(1,2,0),(0,1,2)\}$.
   (Answer: $\mathcal{B}' = \{(1/\sqrt{2}, 1/\sqrt{2}, 0), (-1/\sqrt{2}, 1/\sqrt{2}, 0), (0,0,1)\}$.)

5.2 Exercises

1. *Prove that an orthogonal set of nonzero vectors are linearly independent. (Theorem 5.15.)

2. Apply the Gram-Schmidt orthonormalization process to the basis
   $\mathcal{B} = \{(2, 1, -1), (-2, 1, -3), (1, 1, 1)\}$.
   (Answer: $\mathcal{B}' = \{1/\sqrt{6}(2, 1, -1), 1/\sqrt{14}(-2, 1, -3), 1/\sqrt{21}(-1, 4, , 2)\}$.)

6 Linear Transformations (Chapter 6)

Definition 6.1 *Let V and W be vector spaces. The map $T : V \to W$ is a linear transformation of V into W if:*

(i) $T(u + v) = T(u) + T(v)$ *for all* u, $v \in V$.

(ii) $T(cu) = cT(u)$ *for all* $u \in V$ *and any scalar* c.

Remark 6.2 *Since a linear transformation is a function, we will use the notions of domain, codomain, image of a vector, range and preimage of a vector.*

Theorem 6.3 (Properties) *Let T be a linear transformation from V to W and assume u and v are vectors in V. Then, the following properties hold:*

(i) $T(0) = 0$.

(ii) $T(-v) = -T(v)$.

(iii) $T(u - v) = T(u) - T(v)$.

Example 6.4 *The followings are examples of linear transformations:*

1. *The function $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(v) = Av$ where A is the matrix*

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{pmatrix}.$$

2. *The counterclockwise rotation in $\mathbb{R}^2$ about the origin through an angle θ. This linear transformation can be defined by the matrix:*

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

3. *The function that maps an $n \times m$ matrix to its transpose, i.e., $T : M_{n,m} \to M_{m,n}$ defined by $T(A) = A^T$.*

4. *The differential operator.*

Theorem 6.5 *Let A be an $m \times n$ matrix. The function T defined by $T(v) = Av$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.*

Remark 6.6 *Every linear transformation involving finite-dimensional vector spaces can be represented by matrices.*

6.1 The Kernel and Range

Definition 6.7 *Let $T : V \to W$ be a linear transformation. Then the set of all vectors $v \in V$ that satisfy $T(v) = 0$ is the kernel of T. That is*

$$\ker(T) = \{v \in V \,|\, T(v) = 0\} .$$

Example 6.8 *Find the kernel of:*

1. *The projection $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x, y, 0)$.*
   (Answer: $\ker(T) = \{(0, 0, a) \in \mathbb{R}^3 \,|\, a \in \mathbb{R}\}$.)

2. *The linear transformation T defined by the matrix*

$$A = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{pmatrix} .$$

   (Answer: $\ker(T) = \{(a, -a, a) \in \mathbb{R}^3 \,|\, a \in \mathbb{R}\}$.)

Theorem 6.9 *The kernel of a linear transformation $T : V \to W$ is a subspace of the domain V.*

Theorem 6.10 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation $T(x) = Ax$. Then the kernel of T is the solution space of $Ax = 0$.*

Definition 6.11 *The range of a linear transformation $T : V \to W$ is the set of all vectors $w \in W$ that are images of vectors in V. That is,*

$$\text{range}(T) = \{T(v) \,|\, v \in V\} .$$

Theorem 6.12 *The range of a linear transformation $T : V \to W$ is a subspace of W.*

Definition 6.13 *Let $T : V \to W$ be a linear transformation. The dimension of the kernel of T is called the nullity of T, while the dimension of the range of T is called the rank of T.*

Theorem 6.14 *Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V into a vector space W. Then,*

$$\dim(\ker(T)) + \dim(\text{range}(T)) = n .$$

Definition 6.15 *A function $T : V \to W$ is injective when the preimage of every w in the range consists of a single vector. It is said to be surjective when every element in W has a preimage in V. A function that is both injective and surjective is bijective.*

Definition 6.16 *A linear transformation $T : V \to W$ that is bijective is called an* <u>*isomorphism.*</u> *If there exist an isomorphism between two vector spaces V and W, they are said to be* <u>*isomorphic,*</u> *$V \cong W$.*

Theorem 6.17 *Two finite-dimensional vector spaces V and W are isomorphic if and only if they have the same dimension.*

Example 6.18 *The following vector spaces are isomorphic:*

1. $\mathbb{R}^{n^2}$.
2. *The space of square matrices of order n.*
3. *The space of all polynomials of degree at most $n^2 - 1$.*

6.2 Matrices for Linear Transformations

Remark 6.19 *Every linear transformation involving finite-dimensional vector spaces can be represented by matrices.*

Definition 6.20 *Let V and W be finite-dimensional vector spaces with bases $\mathcal{B} = \{v_1, ..., v_n\}$ and $\mathcal{B}' = \{u_1, ..., u_m\}$, respectively. If $T : V \to W$ is a linear transformation such that for every $i = 1, ..., n$,*

$$T(v_i)_{\mathcal{B}'} = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix},$$

is the coordinate matrix of $T(v_i)$ with respect to the basis $\mathcal{B}'$, then the $m \times n$ matrix

$$A_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix},$$

is the representation of T with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$.

Definition 6.21 *If the bases $\mathcal{B}$ and $\mathcal{B}'$ are the standard bases, then $A_{\mathcal{B}\mathcal{B}'} \equiv A$ is the* <u>*standard*</u> *matrix representation of T.*

Example 6.22 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(x, y) = (x+y, 2x-y)$.*

1. *Find the matrix for T relative to the bases $\mathcal{B} = \{(1, 2), (-1, 1)\}$ and $\mathcal{B}' = \{(0, 1), (1, 1)\}$.*
   Answer:
   $$A_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} -3 & -3 \\ 3 & 0 \end{pmatrix}.$$

2. *Find the image of the vector* $v = (2, 1)$.
Answer:

$$T(v) = (3, 3)\,, \quad T(v)_{\mathcal{B}'} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Remark 6.23 *It is possible to modify the matrix representing* T *so that it changes the bases. This is done with the transition matrices.*

Proposition 6.24 *Let* V *be a vector space with bases* $\mathcal{B} = \{v_1, ..., v_n\}$ *and* $\bar{\mathcal{B}} = \{\bar{v}_1, ..., \bar{v}_n\}$*, and* W *be the vector space with bases* $\mathcal{B}' = \{u_1, ..., u_m\}$ *and* $\bar{\mathcal{B}}' = \{\bar{u}_1, ..., \bar{u}_m\}$*. Denote by* P *the transition matrix from* $\mathcal{B}$ *to* $\bar{\mathcal{B}}$ *and by* Q *the transition matrix from* $\mathcal{B}'$ *to* $\bar{\mathcal{B}}'$*. If* A *is the representation matrix of the linear transformation* $T : V \to W$ *with respect to the bases* $\mathcal{B}$ *and* $\mathcal{B}'$*, then*

$$\bar{A} = QAP^{-1}\,,$$

is the representation matrix of T *with respect to the bases* $\bar{\mathcal{B}}$ *and* $\bar{\mathcal{B}}'$*.*

Example 6.25 *Compute the representation matrices of the linear transformations:*

1. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be the linear transformation defined by* $T(x, y) = (2x - 2y, -x + 3y)$*. Compute the representation matrix* $A_{\mathcal{B}'\mathcal{B}'}$ *where* $\mathcal{B}' = \{(1, 0), (1, 1)\}$*.*
Answer:

$$A_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}.$$

2. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *be the linear transformation defined by* $T(x, y) = (x + y, x - y, x)$*.*

   (a) *Compute the representation matrix* $A = A_{\mathcal{B}\mathcal{B}'}$ *where* $\mathcal{B} = \{(1, 1), (0, 1)\}$ *and* $\mathcal{B}' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$*.*
   Answer:

$$A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

   (b) *Compute the representation matrix* $\bar{A} = A_{\bar{\mathcal{B}}\bar{\mathcal{B}}'}$ *where* $\bar{\mathcal{B}} = \{(1, -1), (1, 0)\}$ *and* $\bar{\mathcal{B}}' = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$*.*
   Answer:

$$\bar{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -2 & 0 \end{pmatrix}.$$

   (c) *Check the relation* $\bar{A} = QAP^{-1}$*.*
   Answer:

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

3. *Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T(x, y, z) = (x+y+z, x-y-z)$. For previous bases:*

   (a) *Compute the representation matrix $A = A_{\mathcal{B}'\mathcal{B}}$.*
   Answer:
   $$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix}.$$

   (b) *Compute the representation matrix $\bar{A} = A_{\bar{\mathcal{B}}'\bar{\mathcal{B}}}$.*
   Answer:
   $$\bar{A} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \end{pmatrix}.$$

   (c) *Check the relation $\bar{A} = QAP^{-1}$.*
   Answer:
   $$Q = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

   Note that these Q and P^{-1} are just the inverses of P^{-1} and Q of the previous problem, respectively. It is just a different choice of name for them.

Remark 6.26 *If $V = W$ one can use the same bases and above formula reads $\bar{A} = PAP^{-1}$.*

Definition 6.27 *Two squares matrices of the same order A and $\bar{A}$ are said to be similar when there exists an invertible matrix P such that $\bar{A} = PAP^{-1}$.*

6.3 Exercises

1. Show that the differential operator is a linear transformation.
2. Let $D_x : P_2 \to P_1$ be the differential operator that maps a polynomial of degree 2 or less into its derivative. Find the matrix for D_x using the bases $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{B}' = \{x, 1\}$.
3. Prove Theorem 6.17.

7 Eigenvalues and Eigenvectors (Chapter 7)

Definition 7.1 *Let A be a square matrix of order n. A scalar λ is an eigenvalue of A when there is a nonzero vector v such that $Av = \lambda v$. The vector v is an eigenvector of A corresponding to λ.*

Example 7.2 *Consider the matrix*

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Verify that the vectors $v = (-3, -1, 1)$ and $u = (1, 0, 0)$ are eigenvectors of A and compute the corresponding eigenvalues.

Theorem 7.3 *Let A be a square matrix of order n with an eigenvalue λ. Then the set of all eigenvectors of λ, together with the zero vector, is a subspace of $\mathbb{R}^n$. This subspace is called the eigenspace of λ.*

Proposition 7.4 *Let A be a square matrix of order n. Then:*

(i) The eigenvalues of A are the scalars λ solution of $|A - \lambda I_n| = 0$.

(ii) The eigenvectors of A corresponding to λ are the nonzero solutions of $(A - \lambda I_n)v = 0$.

Definition 7.5 *The equation $|A - \lambda I_n| = 0$ is called the characteristic equation of A. Moreover, the polynomial $|A - \lambda I_n|$ is the characteristic polynomial of A.*

Remark 7.6 *The eigenvalues are the roots of the characteristic polynomial.*

Example 7.7 *Find the eigenvalues and eigenvectors of the following matrices:*

1.
$$A = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}.$$

(Answer: $\lambda_1 = -1$, $v_1 = (4a, a)$, and $\lambda_2 = -2$, $v_2 = (3a, a)$, for $a \in \mathbb{R}$.)

2.
$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(Answer: $\lambda = 2$ and $v = (a, 0, b)$, for $a, b \in \mathbb{R}$.)

3.

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}.$$

(Answer: $\lambda_1 = 1$, $v_1 = (2a, b, -2a, -a)$; $\lambda_2 = 2$, $v_2 = (0, 5a, a, 0)$, and $\lambda_3 = 3$, $v_3 = (0, 5a, 0, -a)$, for $a, b \in \mathbb{R}$.)

4.

$$D = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(Answer: $\lambda_1 = -2$, $v_1 = (a, -a, b)$, and $\lambda_2 = 4$, $v_2 = (a, a, 0)$, for $a, b \in \mathbb{R}$.)

7.1 Diagonalization

Definition 7.8 *A square matrix A is diagonalizable when it is similar to a diagonal matrix. That is, A is diagonalizable when there exists an invertible matrix P such that PAP^{-1} is diagonal.*

Proposition 7.9 *Two similar matrices have the same eigenvalues.*

Theorem 7.10 *A square matrix of order n is diagonalizable if and only if it has n linearly independent eigenvectors.*

Remark 7.11 *This happens, precisely, when the eigenvalues are real and the dimension of each eigenspace coincides with the multiplicity of the corresponding eigenvalue.*

Remark 7.12 *Let A be a square matrix of order n. In order to diagonalize A:*

(i) We find n linearly independent eigenvectors (whose existence is equivalent to A being diagonalizable).

(ii) We construct the bases of eigenvectors $\mathcal{B}'$ and denote by P the transition matrix from the standard basis to $\mathcal{B}'$.

(iii) Then, PAP^{-1} is the diagonal matrix with the eigenvalues of A in the diagonal.

Example 7.13 *Diagonalize, if possible, the following matrices:*

1.

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix}.$$

Answer: $\mathcal{B}' = \{(1,-1,4),(1,0,-1),(1,-1,-1)\}$ *and*

$$D = A_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P = \frac{1}{5}\begin{pmatrix} 1 & 0 & 1 \\ 5 & 5 & 0 \\ -1 & -5 & -1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 4 & -1 & -1 \end{pmatrix}.$$

2.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}.$$

Answer: $\mathcal{B}' = \{(2,0,-2,-1),(0,1,0,0),(0,5,1,0),(0,5,0,-1)\}$ *and*

$$D = B_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad P = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -5 & 2 & -10 & 10 \\ 2 & 0 & 2 & 0 \\ -1 & 0 & 0 & -2 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 5 & 5 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}.$$

3.

$$C = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix}.$$

Answer: $\mathcal{B}' = \{(5,4,-12),(2,1,0),(1,0,0)\}$ *and*

$$D = C_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \frac{1}{12}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 12 & 4 \\ 12 & -24 & -3 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 5 & 2 & 1 \\ 4 & 1 & 0 \\ -12 & 0 & 0 \end{pmatrix}.$$

Example 7.14 *Diagonalize the following linear transformations:*

1. *The linear transformation* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $T(x,y,z) = (x-y-z, x+3y+z, -3x+y-z)$.
   Answer: $\mathcal{B}' = \{(1,0,1),(1,1,0),(1,5,1)\}$ *and*

$$D_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P = \frac{1}{5}\begin{pmatrix} 1 & -1 & 4 \\ 5 & 0 & -5 \\ -1 & 1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 1 & 0 & 1 \end{pmatrix}.$$

2. *The linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $T(x,y) = (x+y, x+y)$.
   Answer: $\mathcal{B}' = \{(1,-1),(1,1)\}$ *and*

$$D_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad P = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

3. *The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (-2x + 2y - 3z, 2x + y - 6z, -x - 2y)$.*
   Answer: $\mathcal{B}' = \{(1, 0, 1), (0, 1, 2), (1, 2, -3)\}$ *and*

$$D_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad P = \frac{1}{8} \begin{pmatrix} 7 & -2 & 1 \\ -2 & 4 & 2 \\ 1 & 2 & -1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & -3 \end{pmatrix}.$$

7.2 Symmetric Matrices

Definition 7.15 *A square matrix A is symmetric when it is equal to its transpose, i.e., $A^T = A$.*

Definition 7.16 *A square matrix A is orthogonal when it is invertible and $A^{-1} = A^T$.*

Proposition 7.17 *A square matrix A is orthogonal if and only if its column vectors form an orthonormal set (with respect to the standard inner product).*

Theorem 7.18 (Fundamental Theorem of Symmetric Matrices) *Let A be a square matrix. Then A is orthogonally diagonalizable (and has real eigenvalues) if and only if A is symmetric.*

Remark 7.19 *Let A be a symmetric square matrix of order n. In order to diagonalize A:*

(i) We find n linearly independent eigenvectors (whose existence is equivalent to A being diagonalizable).

(ii) We make sure they are an orthonormal set. Eigenspaces of different eigenvalues are orthogonal. In each eigenspace, if needed, we may use Gram-Schmidt.

(iii) We construct the bases of orthonormal eigenvectors $\mathcal{B}'$ and denote by P the transition matrix from the standard basis to $\mathcal{B}'$.

(iv) Then, $PAP^{-1} = PAP^T$ is the diagonal matrix with the eigenvalues of A in the diagonal.

Example 7.20 *Diagonalize the following matrices using orthogonal transition matrices:*

1.
$$A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}.$$

   Answer: $\mathcal{B}' = \{1/\sqrt{5}(2, -1), 1/\sqrt{5}(1, 2)\}$ *and*

$$D = A_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}, \quad P^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \quad P = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

2.

$$B = \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}$$

Answer: $\mathcal{B}' = \{1/3(1,-2,2), 1/\sqrt{5}(2,1,0), 1/3\sqrt{5}(2,-4,-5)\}$ *and*

$$D = B_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$P^T = \frac{1}{3\sqrt{5}} \begin{pmatrix} \sqrt{5} & 6 & 2 \\ -2\sqrt{5} & 3 & -4 \\ 2\sqrt{5} & 0 & -5 \end{pmatrix}, \qquad P = \frac{1}{3\sqrt{5}} \begin{pmatrix} \sqrt{5} & -2\sqrt{5} & 2\sqrt{5} \\ 6 & 3 & 0 \\ 2 & -4 & -5 \end{pmatrix}.$$

7.3 Exercises

1. Diagonalize the following matrices:

   (a)

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$$

   Answer: $\mathcal{B}' = \{(1,1,0), (1,0,-1), (1,0,1)\}$ and

$$D = A_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad P = \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

   (b)

$$B = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

   Answer: $\mathcal{B}' = \{(1,1,0), (1,0,-1), (1,1,2)\}$ and

$$D = B_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad P = \frac{1}{2} \begin{pmatrix} -1 & 3 & -1 \\ 2 & -2 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}.$$

2. Diagonalize the following linear transformations:

   (a) $T(x,y,z) = (3x - y + z, -2x + 4y - 2z, -2x + 2y)$.
   Answer: $\mathcal{B}' = \{(1,1,0), (1,0,-1), (1,-2,-2)\}$ and

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 2 & -1 & 2 \\ -2 & 2 & -3 \\ 1 & -1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & -1 & -2 \end{pmatrix}.$$

(b) $T(x, y, z) = (z, y, 4x)$.

Answer: $\mathcal{B}' = \{(1, 0, -2), (0, 1, 0), (1, 0, 2)\}$ and

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad P = \frac{1}{4}\begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \end{pmatrix}.$$

3. * Find the value of $a \in \mathbb{R}$ such that the matrix

$$A = \begin{pmatrix} 1 & a & a \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

is diagonalizable. For that value of $a \in \mathbb{R}$ compute A^n for an arbitrary $n \in \mathbb{N}$.

Answer: In order the matrix A to be diagonalizable $a = 0$ must hold. In this case, $\mathcal{B}' = \{(1, 0, -1), (0, 1, 0), (0, 1, -1)\}$ and

$$D = A_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}.$$

Finally,

$$A^n = \begin{pmatrix} 1 & 0 & 0 \\ 1 - 2^n & 1 & 1 - 2^n \\ 2^n - 1 & 0 & 2^n \end{pmatrix}.$$

Review Problems

1. Consider the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x,y,z) = (2x+y+z, x+2y+z, x+y+2z)$.

   (a) Prove that the following sets of vectors are bases for $\mathbb{R}^3$:

   $$\begin{aligned} \mathcal{B} &= \{(1,0,1),(1,1,0),(0,1,1)\}, \\ \widetilde{\mathcal{B}} &= \{(1,1,1),(1,0,0),(0,0,-1)\}. \end{aligned}$$

   (b) Compute the representation matrix of the linear transformation T with respect to the bases $\mathcal{B}$ and $\widetilde{\mathcal{B}}$, $A_{\mathcal{B}\widetilde{\mathcal{B}}}$. (Use the definition of $T(x,y,z)$.)
   Answer:

   $$A_{\mathcal{B}\widetilde{\mathcal{B}}} = \begin{pmatrix} 2 & 3 & 3 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

   (c) Compute the representation matrix of the linear transformation T with respect to the bases $\mathcal{B}$ and $\widetilde{\mathcal{B}}$, $A_{\mathcal{B}\widetilde{\mathcal{B}}}$. (Use the representation matrix $A = A_{\mathcal{B}_s\mathcal{B}_s}$ with respect to the standard/canonical bases and the product with transition matrices.)
   Answer: $A_{\mathcal{B}\widetilde{\mathcal{B}}} = QAP^{-1}$, where

   $$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

   (d) Compute the image of the vector $v = (1,-1,2)$ using the representation matrix $A_{\mathcal{B}\widetilde{\mathcal{B}}}$.
   Answer: $T(v) = (3,1,4)$.

   (e) Obtain the eigenvalues of the representation matrix $A = A_{\mathcal{B}_s\mathcal{B}_s}$ of the linear transformation T and a basis of eigenvectors. Is A diagonalizable? (Justify your answer.)
   Answer: The eigenvalues are $\lambda_1 = 1$ (double) and $\lambda_2 = 4$ (single). And, a basis of eigenvectors is $\mathcal{B}' = \{(1,-1,0),(1,1,-2),(1,1,1)\}$.

   (f) Diagonalize A using orthogonal transition matrices.
   Answer: $\mathcal{B}' = \{1/\sqrt{2}(1,-1,0), 1/\sqrt{6}(1,1,-2), 1/\sqrt{3}(1,1,1)\}$ and

   $$D = A_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

   $$P = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix}, \qquad P^T = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \end{pmatrix}.$$

2. Consider the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x,y,z) = (3x-2y+4z, -2x+6y+2z, 4x+2y+3z)$.

(a) Prove that the following set of vectors is a basis for $\mathbb{R}^3$:

$$\mathcal{B} = \{(1,0,1),(1,-1,0),(0,1,-1)\}\,.$$

(b) Compute the representation matrix of the linear transformation T with respect to the basis $\mathcal{B}$, $A_{\mathcal{B}\mathcal{B}}$. (Use the definition of $T(x,y,z)$.)
Answer:

$$A_{\mathcal{B}\mathcal{B}} = \frac{1}{2}\begin{pmatrix} 14 & -1 & -3 \\ 0 & 11 & -9 \\ 0 & -5 & -1 \end{pmatrix}.$$

(c) Compute the representation matrix of the linear transformation T with respect to the bases $\mathcal{B}$, $A_{\mathcal{B}\mathcal{B}}$. (Use the representation matrix $A = A_{\mathcal{B}_s\mathcal{B}_s}$ with respect to the standard/canonical bases and the product with transition matrices.)
Answer: $A_{\mathcal{B}\widetilde{\mathcal{B}}} = PAP^{-1}$, where

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

(d) Compute the image of the vector $v = (-1,-1,0)$ using the representation matrix $A_{\mathcal{B}\mathcal{B}}$.
Answer: $T(v) = (-1,-4,-6)$.

(e) Obtain the eigenvalues of the representation matrix $A = A_{\mathcal{B}_s\mathcal{B}_s}$ of the linear transformation T and a basis of eigenvectors. Is A diagonalizable? (Justify your answer.)
Answer: The eigenvalues are $\lambda_1 = -2$ (single) and $\lambda_2 = 7$ (double). And, a basis of eigenvectors is $\mathcal{B}' = \{(2,1,-2),(1,-2,0),(4,2,5)\}$.

(f) Diagonalize A using orthogonal transition matrices.
Answer: $\mathcal{B}' = \{1/3(2,1,-2), 1/\sqrt{5}(1,-2,0), 1/3\sqrt{5}(4,2,5)\}$ and

$$D = A_{\mathcal{B}'\mathcal{B}'} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix},$$

$$P = \frac{1}{3\sqrt{5}}\begin{pmatrix} 2\sqrt{5} & \sqrt{5} & -2\sqrt{5} \\ 3 & -6 & 0 \\ 4 & 2 & 5 \end{pmatrix}, \qquad P^T = \frac{1}{3\sqrt{5}}\begin{pmatrix} 2\sqrt{5} & 3 & 4 \\ \sqrt{5} & -6 & 2 \\ -2\sqrt{5} & 0 & 5 \end{pmatrix}.$$

(g) Obtain the eigenvalues of the representation matrix $A_{\mathcal{B}\mathcal{B}}$ of the linear transformation T and a basis of eigenvectors. Is it diagonalizable? (Justify your answer.)

(h) Diagonalize it.

8 Appendix A. Linear Congruences

Definition 8.1 *A linear equation $a_1x_1 + ... + a_nx_n = b$ with $a_1, ..., a_n, b \in \mathbb{Z}$ integers, is known as a linear Diophantine equation.*

Remark 8.2 *The purpose of a Diophantine equation is to obtain all integer solutions, that is, all the solutions $s_1, ..., s_n \in \mathbb{Z}$. Of course, if such solutions exist. We will focus on the case of just two variables and write the linear Diophantine equation as*

$$ax + by = c\,,$$

where $a, b, c \in \mathbb{Z}$ are integers.

Proposition 8.3 *The linear Diophantine equation $ax + by = c$ has integer solutions if and only if the greatest common divisor of a and b, $\gcd(a, b)$, divides c.*

Remark 8.4 *Under the assumption that the $\gcd(a, b)$ divides c, we will reduce the Diophantine equation by dividing everything by $\gcd(a, b)$. This will give us an equivalent Diophantine equation*

$$\bar{a}x + \bar{b}y = \bar{c}\,,$$

where now $\gcd(\bar{a}, \bar{b}) = 1$. We will refer to it as the reduced linear Diophantine equation.

Remark 8.5 *Linear Diophantine equations of the type $x + by = c$ can be rewritten in terms of congruences as*

$$x \equiv c \,(\operatorname{mod} b)\,,$$

which means that $x - c$ is an integer multiple of b. In other words, the remainder of dividing x by b is the same as the remainder of dividing c by b.

Definition 8.6 *An integer $a \in \mathbb{Z}$ is said to be congruent to $b \in \mathbb{Z}$ modulo $c \in \mathbb{Z}$, if $a - b$ is an integer multiple of c. We will write*

$$a \equiv b \,(\operatorname{mod} c)\,.$$

Example 8.7 *Check that the following congruences are true.*

1. $416 \equiv 3 \,(\operatorname{mod} 7)$.
2. $22 \equiv -1040 \,(\operatorname{mod} 18)$.

Remark 8.8 *To solve a reduced linear Diophantine equation $ax + by = c$ (or, equivalently, the linear congruence $ax \equiv c \,(\operatorname{mod} b)$) we need to find the inverse of a, that is a^{-1}, modulo b. This requires applying the extended Euclidean algorithm to get Bezout's identity. Then, we have*

$$x \equiv a^{-1}c \,(\operatorname{mod} b)\,,$$

from which we obtain all the values of x. Finally, substituting in the equation $ax + by = c$, we get the values of y. (Note that a^{-1} is the inverse of a modulo b, not $1/a$.)

Example 8.9 *Solve the following linear Diophantine equations:*

1. $3x + 7y = 1$. *(Answer:* $x = -2 + 7n$ *and* $y = 1 - 3n$*, where* $n \in \mathbb{Z}$*.)*

2. $7x + 18y = 208$. *(Answer:* $x = 22 - 18n$ *and* $y = 3 + 7n$*, where* $n \in \mathbb{Z}$*.)*

3. $12x + 8y = 68$. *(Answer:* $x = 1 + 2n$ *and* $y = 7 - 3n$*, where* $n \in \mathbb{Z}$*.)*

Remark 8.10 *We will now consider systems of linear congruences, which arise in problems of the following type: "On a pirate ship there are* 17 *pirates who just stole a chest of gold coins. They try to divide these coins equally among the* 17 *pirates, but there are* 3 *left over. The pirates begin a fight and one of them dies. Once this pirate has died, the others calm down and try to divide all the gold coins equally again. Unfortunately, now there are* 10 *left over coins. Another fight begins and another pirate dies. After this new death, the pirates that are still alive try to divide the coins equally yet again. This time it is possible and there are no left over coins. What is the minimum possible amount of gold coins that the pirates stole?"*

Theorem 8.11 (Chinese Remainder Theorem) *Consider the linear system of congruences*

$$\begin{cases} x & \equiv a_1 \pmod{n_1} \\ & \vdots \\ x & \equiv a_k \pmod{n_k} \end{cases},$$

where $n_1, ..., n_k$ *are pairwise coprime integers. Then, the system has a solution which is unique modulo* $n_1 \cdots n_k$*.*

Example 8.12 *Solve the following systems of linear congruences:*

1. $$\begin{cases} x & \equiv 1 \pmod{3} \\ x & \equiv 4 \pmod{5} \\ x & \equiv 6 \pmod{7} \end{cases}.$$

   (Answer: $x = 34 + 105n$*, where* $n \in \mathbb{Z}$*.)*

2. $$\begin{cases} x & \equiv 3 \pmod{17} \\ x & \equiv 10 \pmod{16} \\ x & \equiv 0 \pmod{15} \end{cases}.$$

   (Answer: $x = 3930 + 4080n$*, where* $n \in \mathbb{Z}$*.)*

3. $$\begin{cases} x & \equiv 7 \pmod{12} \\ x & \equiv 7 \pmod{20} \\ x & \equiv 7 \pmod{36} \end{cases}.$$

   (Answer: $x = 7 + 180n$*, where* $n \in \mathbb{Z}$*.)*

9 Appendix B. Crystallographic Groups and Tessellations

Definition 9.1 *An isometry is a bijective distance-preserving transformation.*

Remark 9.2 *Roughly speaking, an isometry is a motion that may or may not preserve the orientation.*

Proposition 9.3 *In the plane $\mathbb{R}^2$, there are three different types of isometries:*

1. *Translations. A translation is a transformation that moves every point the same distance in a given direction.*
2. *Rotations. A rotation is a motion that preserves a fixed points and moves the rest a given angle.*
3. *Reflections. A reflection is a motion that preserves a line of fixed points (the axis).*

Definition 9.4 *A motion that consists of a reflection over a line and then a translation along that line is usually called a glide-reflection.*

Remark 9.5 *Translations and rotations preserve the orientation, while reflections do not.*

Proposition 9.6 *Isometries of the plane can be represented using matrices.*

Definition 9.7 *A symmetry of a figure is an isometry under which the figure is invariant (remains unchanged).*

Definition 9.8 *A symmetry group is the set of all symmetries of a shape together with the operation of composition of functions.*

Definition 9.9 *A (planar) tessellation (or, tiling) is the covering of the plane (or a part of it sufficiently big) using one or more geometric shapes (tiles) with no overlaps and no gaps.*

Definition 9.10 *A periodic tessellation (or, wallpaper pattern) is a tessellation that has translation symmetries in two linearly independent directions.*

Remark 9.11 *A periodic tessellation is a repeating pattern when tiling the plane.*

Remark 9.12 *If a tessellation has translational symmetries in just one direction, we have a frieze pattern. The symmetry group of a frieze pattern is a frieze group. There are exactly seven essentially different frieze groups.*

Example 9.13 *Let $n > 2$ be a natural number. Consider a regular polygon of n sides $\mathcal{P}_n$ and assume that $l > 0$ is a fixed positive number representing the length of each side of $\mathcal{P}_n$.*

1. *Compute the perimeter of $\mathcal{P}_n$ for every $n > 2$. (Answer: $l \cdot n$.)*
2. *Compute the area of $\mathcal{P}_n$ for every $n > 2$. (Answer: $nl^2/(4\tan(\pi/n))$.)*
3. *Compute the angle between any two sides of $\mathcal{P}_n$ for every $n > 2$. (Answer: $(n-2)\pi/n$.)*
4. *Find all possible regular polygons which can tessellate the plane. (Answer: Triangles, Squares and Hexagons.)*
5. *Among these possible polygons, find the one that maximizes the ratio area-length. (Answer: Hexagon.)*

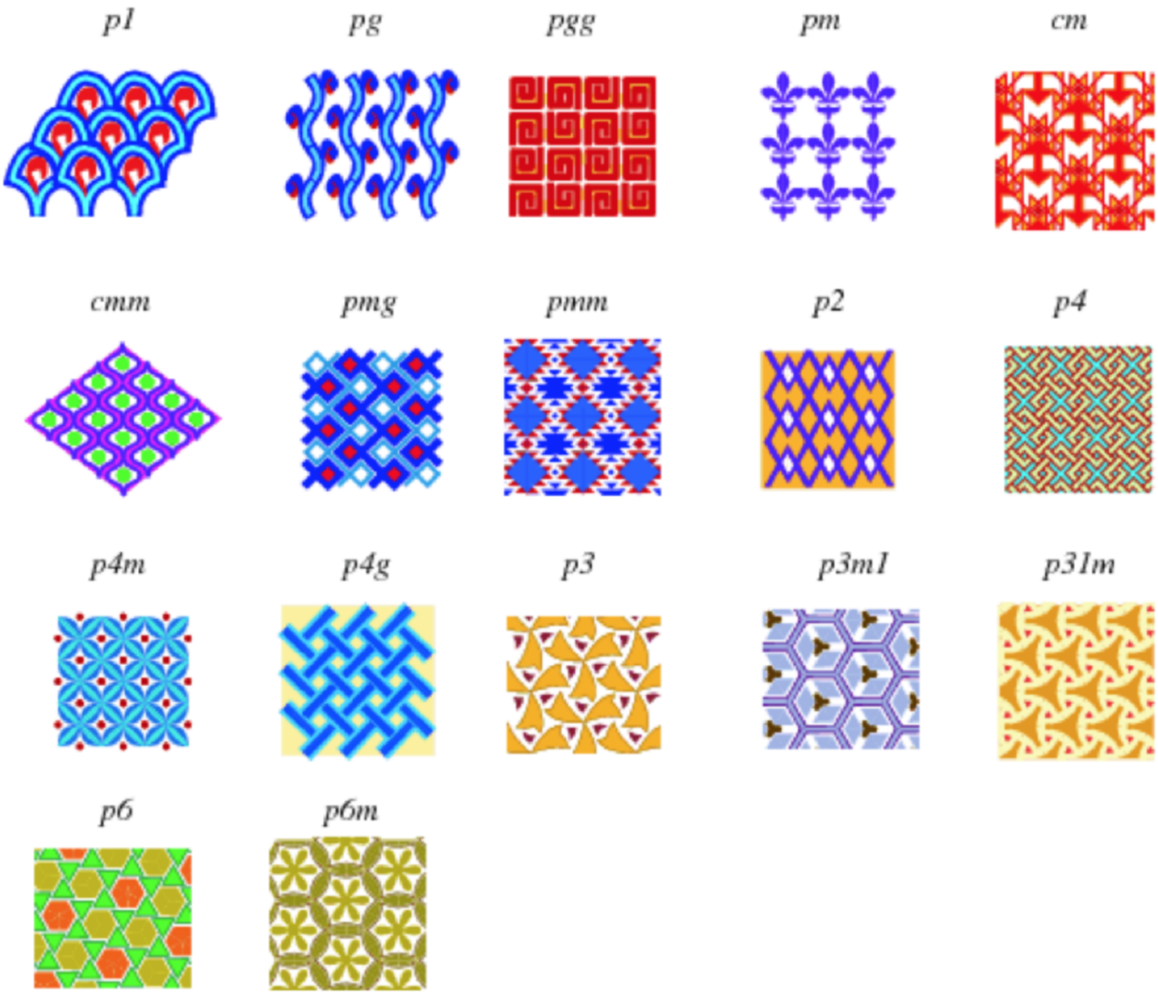
As a consequence, based on the Principle of Least Action that governs Nature, bees make honey using beehives with the shape of regular hexagons.

Definition 9.14 *A wallpaper group (or, planar crystallographic group) is the symmetry group of a periodic tessellation.*

Remark 9.15 *A crystal or crystalline solid is a solid material whose constituents, such as atoms, molecules, or irons, are arranged in a highly ordered microscopic structure. The patterns that crystal forms have symmetry groups called crystallographic groups.*

Theorem 9.16 *There are exactly 17 essentially different wallpaper groups.*

Remark 9.17 *Apparently, the 17 groups are represented at La Alhambra.*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