LECTURE NOTES

Math 3350, Ordinary Differential Equations

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1 Introduction (Chapter 1)

Definition 1.1 *Any equation involving ordinary or partial derivatives is called a differential equation.*

Definition 1.2 *A differential equation is an ordinary differential equation (ODE) if it has only ordinary derivatives in it. To the contrary, if it has partial derivatives in it, is called a partial differential equation (PDE).*

Remark 1.3 *Throughout this course we will only consider ordinary differential equations.*

Example 1.4 *Newton's second law of motion states that, at any instant of time t, the total force F acting on a body is equal to the body's acceleration times the mass m. This physical law can be mathematically modeled by*

$$F = m\frac{d^2y}{dt^2} = my''\,,$$

where $y(t)$ is the position of the object.

Definition 1.5 *The order of a differential equation is the order of the highest derivative that appears in the equation.*

Definition 1.6 *A linear differential equation is a differential equation that can be written as*

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t)\,,$$

where the coefficients $a_0(t),\ldots,a_n(t)$ and $g(t)$ can be linear or nonlinear functions.

If a differential equation cannot be written in the above form, then it is a nonlinear differential equation.

Definition 1.7 *A solution to an ordinary differential equation on an interval* $\alpha < t < \beta$ *is any function* $y(t)$ *which satisfies the equation.*

Definition 1.8 *Given an ordinary differential equation, the initial conditions is a set of conditions on the solution that specify which particular solution we are looking for.*

Remark 1.9 *The initial conditions are values of the solution or its derivatives at specific points. Roughly speaking, these values allow us to determine the constants of integration.*

Definition 1.10 *An initial value problem is a differential equation with an appropriate number of initial conditions (the number of necessary initial conditions depends on the order of the equation itself).*

2 First-Order Differential Equations (Chapter 2)

Remark 2.1 *The main goal of this section is to solve ordinary differential equations of the type*

$$\frac{dy}{dt} = y' = f(y,t)\,,$$

which is the general form of first order differential equations.

Theorem 2.2 (Existence and Uniqueness – Linear) *Let p and g be continuous functions on an open interval $\alpha < t < \beta$ containing $t = t_o$. Then, there exists a unique solution of the initial value problem*

$$y' + p(t)y = g(t)\,,$$

with initial condition

$$y(t_o) = y_o\,,$$

where y_o is an arbitrary prescribed function.

Theorem 2.3 (Existence and Uniqueness – Nonlinear) *Let f and $\partial f/\partial y$ be continuous functions in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_o, y_o). Then, in some "small" interval around t_o contained in (α, β), there is a unique solution of the initial value problem*

$$y' = f(y,t)\,, \qquad y(t_o) = y_o\,.$$

Remark 2.4 *Geometrically, the uniqueness part of above results means that the graphs of two solutions cannot intersect each other.*

Remark 2.5 *Unfortunately, for an arbitrary function f, there is no general method for solving the first order differential equation $y' = f(y,t)$. Hence, we will next describe different methods for different sub-classes of equations.*

2.1 Linear Equations – Method of Integrating Factors

Remark 2.6 *Consider a first order linear differential equation of the form*

$$\frac{dy}{dt} + p(t)y = g(t)\,.$$

If our equation is not already in this form we need to rewrite it.

The idea to find a solution is to multiply the equation by a function $\mu(t)$ (to be determined) such that the left-hand side becomes the derivative of the product $\mu(t)y(t)$. The unknown function $\mu(t)$ is the <u>integrating factor</u>.

If $\mu(t)$ is a solution of

$$\mu'(t) = p(t)\mu(t)\,,$$

then we have that the left-hand side is $(\mu(t)y(t))'$, as we wanted.

Hence, we just need to solve for $\mu(t)$ above (after a simple manipulation dividing by $\mu(t)$ and direct integration to obtain the logarithm), that is,

$$\mu(t) = e^{\int p(t)\,dt}\,.$$

Then we integrate the original equation multiplied by $\mu(t)$, to obtain

$$\mu(t)y = \int \mu(t)g(t)\,dt\,,$$

where $\mu(t)$ is given above.

Finally, if necessary, we solve for $y(t)$, to conclude with

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)g(t)\,dt\,.$$

Remark 2.7 *Since we are integrating twice, we may get two different constants of integration. We must combine them in just one unknown constant (to be determined by the initial condition, if any).*

Example 2.8 *Find the general solutions of the following differential equations:*

1. $(4+t^2)y' + 2ty = 4t$.
2. $y' - 2y = 4 - t$.
3. $y' + 3y = t + e^{-2t}$.
4. $y' + y = te^{-t} + 1$.
5. $y' - 2y = 3e^t$.
6. $y' + y = 5\sin(2t)$.
7. $y' - 2y = t^2e^{2t}$.
8. $2y' + y = 3t^2$.

Example 2.9 *Find the solution to the following initial value problems:*

1. $2y' + y = e^{t/3}\,, \qquad y(0) = 1$.
2. $ty' + 2y = 4t^2\,, \qquad y(1) = 2$.
3. $2y' + ty = 2\,, \qquad y(0) = 1$.
4. $y' - y = 2te^{2t}\,, \qquad y(0) = 1$.

5. $y' + 2y = te^{-2t}$, $\qquad y(1) = 0$.

6. $t^2 y' + 2ty = \cos(t)$, $\qquad y(\pi) = 0$.

7. $ty' + (t+1)y = y$, $\qquad y(\log 2) = 1$.

8. $2y' + y = 4\cos(t)$, $\qquad y(0) = -1$.

2.2 Separable Equations

Definition 2.10 *A first order differential equation that can be written in the form*

$$N(y)\frac{dy}{dx} = M(x)\,,$$

is said to be separable.

Remark 2.11 *Observe that we are using x to denote the independent variable (rather than t).*

Remark 2.12 *Roughly speaking, the process to solve separable differential equations is to rewrite them in differential form, namely,*

$$N(y)dy = M(x)dx\,,$$

and integrating (formally) both sides with respect to y and x, respectively.

The integration process will give rise to two different constants of integration, but they can be combined in just one (that is, we can 'forget' about one of them).

This method will end up with an implicit solution of the separable equation, that is,

$$f(x, y) = c\,,$$

for a suitable function $f(x, y)$.

Example 2.13 *Find the general solutions of the following differential equations:*

1. $$\frac{dy}{dx} = \frac{x^2}{1 - y^2}\,.$$

2. $$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}\,.$$

3. $$y' = \frac{x^2}{y}\,.$$

4.

$$\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^{y}} .$$

5.

$$y' + y^2 \sin(x) = 0 .$$

6.

$$y' = \frac{-x}{y} .$$

7.

$$\frac{dy}{dx} = \frac{x^2}{1 + y^2} .$$

8.

$$y' = \cos^2(x) \cos^2(2y) .$$

Example 2.14 *Find the solution to the following initial value problems:*

1.

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)} , \qquad y(0) = -1 .$$

2.

$$y' = (1 - 2x) y^2 , \qquad y(0) = -1/6 .$$

3.

$$y' = \frac{1 - 2x}{y} , \qquad y(1) = -2 .$$

4.

$$x dx + y e^{-x} dy = 0 , \qquad y(0) = 1 .$$

5.

$$y' = \frac{2x}{1 + 2y} , \qquad y(2) = 0 .$$

6.

$$y' = \frac{3x^2 - e^x}{2y - 5} , \qquad y(0) = 1 .$$

7.

$$\sin(2x) dx + \cos(3y) dy = 0 , \qquad y(\pi/2) = \pi/3 .$$

8.

$$y' = 2y^2 + xy^2 , \qquad y(0) = 1 .$$

2.3 Exact Equations

Definition 2.15 *An exact differential equation is such that can be expressed exactly as the derivative of a specific function.*

Theorem 2.16 *Let $M(x,y)$, $N(x,y)$, $M_y(x,y)$, and $N_x(x,y)$ be continuous functions defined on a simply-connected domain $\mathcal{D}$. Then, the differential equation*

$$M(x,y) + N(x,y)y' = 0\,,$$

is an exact differential equation on $\mathcal{D}$ if and only if

$$M_y(x,y) = N_x(x,y)\,,$$

at each point in $\mathcal{D}$.

Remark 2.17 *Consider an exact differential equation*

$$M(x,y) + N(x,y)y' = 0\,.$$

Then, by definition, there exists a function $f(x,y)$ such that

$$\frac{d}{dx}f(x,y) = f_x(x,y) + f_y(x,y)\frac{dy}{dx} = M(x,y) + N(x,y)y' = 0\,.$$

That is,

$$f_x(x,y) = M(x,y)\,, \qquad f_y(x,y) = N(x,y)\,.$$

In order to find the solution, we need to integrate these conditions. This will end up with an implicit solution ($f(x,y) = c$).

Example 2.18 *Find the general solutions of the following differential equations:*

1. $2x + y^2 + 2xyy' = 0\,.$
2. $y\cos x + 2xe^y + (\sin x + x^2e^y - 1)y' = 0\,.$
3. $2xy - 9x^2 + (2y + x^2 + 1)y' = 0\,.$
4. $2xy^2 + 4 = 2(3 - x^2y)y'\,.$
5. $2x + 3 + (2y - 2)y' = 0\,.$
6. $2x + 4y + 2(x - y)y' = 0\,.$
7. $3x^2 - 2xy + 2 + (6y^2 - x^2 + 3)y' = 0\,.$
8. $ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x + (xe^{xy}\cos(2x) - 3)y' = 0\,.$

Example 2.19 *Find the solution to the following initial value problems:*

1. $2x - y + (2y - x)y' = 0\,, \qquad y(1) = 3\,.$
2. $9x^2 + y - 1 - (4y - x)y' = 0\,, \qquad y(1) = 0\,.$

2.4 Modeling with First-Order Differential Equations

Example 2.20 (Mixing Problems) *A lake of capacity* 20 *million gallons initially contains* 10 *million gallons of water with* 3 *lbs of undesirable chemical in it. Water containing this chemical flows into the lake at a rate of* 5 *million gallons per year, and the mixture in the lake flows out at* 1 *million gallons per year. The concentration of the chemical in the incoming water varies periodically with time according to* $2+\sin(2t)$ *lbs/gal. Determine the amount of chemical in the lake when it overflows, assuming the currents of the lake make it a well mixed solution. (Hint: The rate of change of the amount of chemical equals the rate, i.e., flow times concentration, at which the chemical enters the lake minus the rate at which exists the lake.)*

Example 2.21 (Population) *The population of dinosaurs in the Earth was growing at a rate of twice the current population, which we assume that was* 7 *billion. On a given year,* 1 *billion individuals migrated to Mars, another billion died of 'natural' causes, and* 2 *billion of them were killed by a meteorite. With these numbers, did dinosaurs disappear from the Earth? If yes, how long did it take them to disappear?*
(Hint: The rate of change of population equals the rate of growth minus the rate of population leaving the Earth for whatever reason.)

2.5 Exercises

1. Use the uniqueness of solution for a initial value problem of a suitable linear differential equation to prove Euler's formula

$$e^{ix} = \cos x + i \sin x \,.$$

   (Hint: Consider the initial value problem $y'(x) - iy(x) = 0$, $y(0) = 1$. The imaginary unit i can be treated as a constant number.)

2. A function $f(x, y)$ is called <u>homogeneous</u> (of degree k) if

$$f(ax, ay) = a^k f(x, y)\,,$$

   for every $a \neq 0$ and some integer k.

   A first-order ordinary differential equation

$$M(x, y) + N(x, y)y' = 0\,,$$

   where $M(x, y)$ and $N(x, y)$ are homogeneous of the same degree can be converted into a separable equation by applying the substitution $u = y/x$.

   Use this substitution, if possible, to solve the following initial value problems:

   (a) $xyy' + 4x^2 + y^2 = 0\,, \qquad y(2) = -7\,.$

   (b) $xy' = y\,(\log x - \log y)\,, \qquad y(1) = 4\,.$

3. Find the solution to the initial value problem

$$y' - (4x - y + 1)^2 = 0\,, \qquad y(0) = 2\,.$$

(Hint: Use the change of variable $u = 4x - y$ to transform this differential equation into a separable one.)

4. Find the solution to the initial value problem

$$y' = e^{9y-x}\,, \qquad y(0) = 0\,.$$

(Hint: Use the change of variable $u = 9y - x$ to transform this differential equation into a separable one.)

3 Second-Order Linear Differential Equations (Chapter 3)

Remark 3.1 *The goal of this section is to introduce the theory associated to second-order linear differential equations. The general form of these equations is*

$$y'' + p(t)y' + q(t)y = g(t)\,.$$

In addition, we will solve those second-order linear differential equations of type

$$y'' + ay' + by = g(t)\,,$$

where a and b are real constants.

Definition 3.2 *A second-order linear differential equation*

$$y'' + p(t)y' + q(t)y = g(t)\,,$$

is said to be <u>homogeneous</u> if the function $g(t)$ is identically zero.

Otherwise, the equation is called <u>nonhomogeneous</u>.

Remark 3.3 *The general solution of a second-order linear differential equation will be composed by the solutions of the associated homogeneous equation and a particular solution of the nonhomogeneous one. Hence, we focus first on the theory of solutions for homogeneous differential equations.*

Theorem 3.4 (Principle of Superposition) *Let y_1 and y_2 be two solutions of the second-order linear homogeneous differential equation*

$$y'' + p(t)y' + q(t)y = 0\,.$$

Then, the linear combination $c_1y_1 + c_2y_2$ (where $c_1, c_2 \in \mathbb{R}$ are real numbers) is also a solution of the differential equation.

Definition 3.5 *Let y_1 and y_2 be two solutions of a second-order linear homogeneous differential equation. The <u>Wronskian</u> of y_1 and y_2 is defined by*

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2\,.$$

The solutions y_1 and y_2 are said to form a <u>fundamental set of solutions</u> if their Wronskian is nonzero.

Remark 3.6 *Given a fundamental set of solutions y_1 and y_2, we can use the principle of superposition to obtain all the solutions to the homogeneous equation. Therefore, the general solution of a homogeneous equation is*

$$y = c_1 y_1 + c_2 y_2 \,,$$

where $c_1, c_2 \in \mathbb{R}$ are constants.

Theorem 3.7 (Abel's Formula) *The Wronskian of two solutions y_1 and y_2 of the second-order linear homogeneous differential equation*

$$y'' + p(t)y' + q(t)y = 0 \,,$$

is given by

$$W[y_1, y_2] = ce^{-\int p(t)dt} \,,$$

where c is a constant (depending on y_1 and y_2).

Consequently, either $W[y_1, y_2]$ is always zero (when $c = 0$) or else is never zero (when $c \neq 0$).

Theorem 3.8 (General Solution) *The general solution to the second-order linear differential equation*

$$y'' + p(t)y' + q(t)y = g(t) \,,$$

can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t) \,,$$

where c_1 and c_2 are real constants, $y_p(t)$ is a particular solution, and y_1 and y_2 form a fundamental set of solutions to the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0 \,.$$

Remark 3.9 *The above result shows that in order to find the general solution, we need to:*

(i) Find the general solution to the associated homogeneous equation.

(ii) Find a particular solution of the nonhomogeneous equation.

(iii) Form the sum of the statement.

Moreover, it also shows that the general solution depends on two parameters (namely, c_1 and c_2). Hence, in order to have a well-posed initial value problem we need to fix two initial conditions.

Theorem 3.10 (Existence and Uniqueness) *Let p, q, and g be continuous functions on an open interval $\alpha < t < \beta$ containing $t = t_o$. Then, there exists a unique solution of the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t) \,,$$

with initial conditions

$$y(t_o) = y_o \,, \qquad \text{and} \qquad y'(t_o) = y'_o \,,$$

where y_o and y'_o are arbitrary prescribed functions.

Remark 3.11 *From now on, we will restrict ourselves to second-order linear differential equations with constant coefficients. That is,*

$$y'' + ay' + by = g(t)\,,$$

where $a, b \in \mathbb{R}$ are constants.

3.1 Homogeneous Equations

Remark 3.12 *Consider the second-order linear homogeneous differential equation with constant coefficients*

$$y'' + ay' + by = 0\,.$$

A natural type of functions to test are exponentials $y(t) = e^{rt}$ where r is a parameter to be determined. This gives rise to the characteristic equation

$$r^2 + ar + b = 0\,.$$

Depending on the type of roots of this quadratic equation, we have three different cases:

(i) Two different real roots. Let $r_1 \neq r_2$ be two real roots of the characteristic equation. Then, the general solution to the original differential equation is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}\,.$$

(ii) Complex (and conjugate) roots. Let $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$ be the two solutions of the characteristic equation. Then, the general solution to the original differential equation is

$$y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)\,.$$

(iii) One (real) double root. Let $r = -a/2$ be the double root of the characteristic equation. Then, the general solution to the original differential equation is

$$y(t) = c_1 e^{-at/2} + c_2 t e^{-at/2}\,.$$

Remark 3.13 *For the second case above, we are using Euler's formula. That is,*

$$e^{(\lambda+i\mu)t} = e^{\lambda} e^{i\mu t} = e^{\lambda}\cos(\mu t) + i e^{\lambda}\sin(\mu t)\,,$$

and the fact that the real and imaginary parts of a complex solution to a linear homogeneous differential equation are also solutions of the same equation.

Remark 3.14 *For the third case above, we only have one solution of the differential equation, namely, $y = e^{-at/2}$. Hence, we apply the reduction of order method to obtain the second solution. This consists of considering a potential solution of the type*

$$y(t) = v(t) e^{-at/2}\,,$$

for a certain function $v(t)$ to be determined. We determine it, by substituting $y(t)$ in the differential equation and forcing it to be a solution.

Proposition 3.15 (Reduction of Order) *Let $y_1(t)$ be a nonzero solution of*

$$y'' + p(t)y' + q(t)y = 0\,.$$

Then, $y(t) = v(t)y_1(t)$ is a second solution for a suitable function $v(t)$ to be determined by plugging $y(t)$ in the equation.

Example 3.16 *Find the general solutions of the following differential equations:*

1. $y'' - y = 0\,.$
2. $y'' + 5y' + 6y = 0\,.$
3. $y'' - 2y' - 2y = 0\,.$
4. $y'' + 9y = 0\,.$
5. $y'' + 2y' + 2y = 0\,.$
6. $y'' - 2y' + 6y = 0\,.$
7. $y'' + 4y' + 4y = 0\,.$
8. $2y'' + 2y' + y = 0\,.$

Example 3.17 *Find the solution to the following initial value problems:*

1. $y'' - y = 0\,, \qquad y(0) = 2\,, \quad y'(0) = -1\,.$
2. $y'' + 5y' + 6y = 0\,, \qquad y(0) = 2\,, \quad y'(0) = 3\,.$
3. $4y'' - 8y' + 3y = 0\,, \qquad y(0) = 2\,, \quad y'(0) = 1/2\,.$
4. $4y'' + 4y' + 37y = 0\,, \qquad y(0) = 2\,, \quad y'(0) = 8\,.$
5. $16y'' - 8y' + 145y = 0\,, \qquad y(0) = -2\,, \quad y'(0) = 1\,.$
6. $4y'' - 4y' + y = 0\,, \qquad y(0) = 2\,, \quad y'(0) = 1/3\,.$
7. $y'' - 6y' + 9y = 0\,, \qquad y(0) = 0\,, \quad y'(0) = 2\,.$
8. $9y'' - 12y' + 4y = 0\,, \qquad y(0) = 2\,, \quad y'(0) = -1\,.$

3.2 Nonhomogeneous Equations

Remark 3.18 *To get the general solution to the second-order linear differential equation*

$$y'' + p(t)y' + q(t)y = g(t)\,,$$

we need a particular solution $y_p(t)$*. We will next see two different methods to find this particular solution.*

Remark 3.19 (Method of Undetermined Coefficients) *The method of undetermined coefficients consist of guessing the particular solution for some coefficients that will later on be determined. A 'good' guess for our particular solution would be:*

(i) An exponential function $Ae^{\alpha t}$ *if* $g(t) = e^{\alpha t}$*.*

(ii) A linear combination of trigonometric functions $A\cos(\beta t) + B\sin(\beta t)$ *if* $g(t) = \cos(\beta t)$ *or* $g(t) = \sin(\beta t)$*.*

(iii) A polynomial of degree n *if* $g(t)$ *is such a polynomial itself.*

Moreover, if $g(t)$ *is composed by products or sums of such functions, our guess should also be composed by such operations.*

Remark 3.20 *If our best guess is already one of the fundamental solutions to the associated homogeneous equation, we need to multiply it by* t *(in analogy with the second fundamental solution in the case of double roots). If this is also a fundamental solution, then we must multiply it by* t^2*.*

Example 3.21 *Find a particular solution of the following second-order linear differential equations:*

1. $y'' - 3y' - 4y = 3e^{2t}\,.$
2. $y'' - 3y' - 4y = 2\sin(t)\,.$
3. $y'' - 3y' - 4y = -8e^{t}\cos(2t)\,.$
4. $y'' - 3y' - 4y = 3e^{2t} + 2\sin(t) - 8e^{t}\cos(2t)\,.$
5. $y'' - 3y' - 4y = 2e^{-2t}\,.$
6. $y'' - 4y' - 12y = 3e^{5t} + \sin(2t) + te^{4t}\,.$
7. $y'' - 4y' - 12y = e^{6t}\,.$
8. $y'' + 8y' + 16y = e^{-4t} + (t^2 + 5)e^{-4t}\,.$

Remark 3.22 (Method of Variation of Parameters) *The method of variation of parameters is a general method based on picking up the general solution of the associated homogeneous equation and substituting the constants by functions to be determined.*

More precisely, if $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of the associated homogeneous equation, we consider

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)\,,$$

and impose the restriction

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0\,.$$

Substituting $y(t)$ in the original nonhomogeneous equation would give us the condition

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)\,.$$

Solving the system given by this condition and the above restriction, we have

$$u_1'(t) = -\frac{y_2(t)g(t)}{W[y_1, y_2]}\,, \qquad u_2'(t) = \frac{y_1(t)g(t)}{W[y_1, y_2]}\,.$$

Therefore, integrating we deduce the particular solution

$$y_p(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W[y_1, y_2]}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W[y_1, y_2]}\,dt\,.$$

Example 3.23 *Find a particular solution of the following second-order linear differential equations:*

1. $y'' + 4y = 8\tan(t)\,.$
2. $y'' + y = \tan(t)\,.$
3. $y'' - 5y' + 6y = 2e^t\,.$
4. $y'' - y' - 2y = 2e^{-t}\,.$
5. $4y'' - 4y' + y = 16e^{t/2}\,.$
6. $y'' + 4y' + 4y = t^{-2}e^{-2t}\,.$
7. $y'' - 2y' + y = e^t/(1 + t^2)\,.$
8. $4y'' + y = 2/\cos(t/2)\,.$

3.3 Modeling with Second-Order Differential Equations

Remark 3.24 (Mechanical Vibrations) *We are going to consider the motion of an object of mass m hanging on the end of a vertical spring, when it is acted on by an external force or it is initially displaced.*

Let $u(t)$ (measured positive in the downward direction) denote the displacement of the mass from its equilibrium position at time t. According to Newton's second law of motion

$$mu''(t) = f(t)\,,$$

where u'' is the acceleration of the mass and f is the net force acting on it.

We next consider the separate forces that compose f:

(i) Gravity. The weight $F_g = mg$ acts downward. (Here, g is the acceleration due to gravity, which we take to be constant.)

(ii) Spring Force. From Hooke's law this force must be proportional to the elongation of the spring. That is,

$$F_s = -k\,(L+u)\,,$$

where L is the elongation at equilibrium and k is the spring constant (i.e., the stiffness). The minus sign arises since this force acts upward.

(iii) Damping. The damping or resistive force F_d, which acts in the direction opposite to the motion, will be assumed to be proportional to the speed (this is known as viscous damping). That is,

$$F_d = -\gamma u'\,,$$

where γ is the damping constant.

(iv) External Force. A external force $F(t)$ may be applied downward or upward.

Hence, combining everything we conclude with

$$mu'' = mg - k\,(L+u) - \gamma u' + F\,.$$

Since at equilibrium, gravity and the spring force cancel out, we deduce that $mg = kL$ and so, above differential equation reads

$$mu'' + \gamma u' + ku = F(t)\,.$$

Example 3.25 *A mass of 4 lb stretches a spring 2 in. Suppose that the mass is given an additional 6 in displacement in the positive direction and then released. The mass is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/s. Formulate the initial value problem that governs the motion of the body, and solve it.*
(Answer: For simplicity, take $g = 1$. Then, from the equilibrium condition $mg = kL$, we deduce that $k = 2$. We also have $\gamma = 6/3 = 2$ and so the differential equation governing the motion is $4u'' + 2u' + 2u = 0$. The initial conditions for the problem are: $u(0) = 6$ and $u'(0) = 0$.)

3.4 Exercises

1. Prove the Principle of Superposition (Theorem 3.4).

2. Show using the reduction of order method that if $r = -a/2$ is the double root of the characteristic equation of the differential equation

$$y'' + ay' + by = 0\,,$$

then $y(t) = te^{-at/2}$ is a solution such that, together with $e^{-at/2}$, form a fundamental set of solutions.

3. The Euler equations are the differential equations of the type

$$t^2y'' + \alpha ty' + \beta y = 0\,,$$

where $\alpha, \beta \in \mathbb{R}$ are real constants.

   (a) Use a suitable change of variable to transform the Euler equation into the differential equation with constant coefficients

$$y''(u) + (\alpha - 1)\, y'(u) + \beta y(u) = 0\,.$$

   (Hint: Consider the new variable $u = \log t$, $t > 0$.)

   (b) Use that the natural type of functions to test for linear differential equations with constant coefficients are exponentials $y(u) = e^{ru}$, with r to be determined, to show that the reasonable functions to test for the original Euler equation are powers $y(t) = t^r$.

   (c) Consider solutions of the Euler equation of the form $y(t) = t^r$, $t > 0$, with the parameter r to be determined. Obtain the condition that the parameter r must satisfy, that is the characteristic equation.

   (d) Show that if the characteristic equation has two different real solutions, say $r_1 \neq r_2$, then $y_1(t) = t^{r_1}$ and $y_2(t) = t^{r_2}$ form a fundamental set of solutions.

   (e) Show that if the characteristic equation has a double root, say r, then the first fundamental solution is $y_1(t) = t^r$, while the second fundamental solution is $y_2(t) = t^r \log t$.

   (f) Show that if the characteristic equation has complex solutions, say $r = \lambda + i\mu$ and $r = \lambda - i\mu$, then the fundamental set of solutions is composed by $y_1(t) = t^\lambda \cos(\mu \log t)$ and $y_2(t) = t^\lambda \sin(\mu \log t)$.

   (g) Use the above to obtain the general solution to

$$2t^2y'' + 3ty' - 15y = 0\,.$$

   (h) Use the above to obtain the general solution to

$$t^2y'' - 7ty' + 16y = 0\,.$$

   (i) Use the above to obtain the general solution to

$$t^2y'' + 3ty' + 4y = 0\,.$$

Review Problems

1. Find the solution to the initial value problem

$$y' + \frac{4}{t}\,y = 6t - 5\,, \qquad y(1) = 1\,.$$

2. Find the solution to the initial value problem

$$y' + \tan(t)y = e^{2t}\cos(t)\,, \qquad y(0) = 2\,.$$

3. Find the solution to the initial value problem

$$xy' = \sqrt{1+y^2}\,, \qquad y(2) = 0\,.$$

4. Find the solution to the initial value problem

$$(2y - x)\,y' = y\,, \qquad y(2) = 1\,.$$

5. Consider the second-order differential equation

$$y'' + y' - 2y = te^t\,.$$

   (a) Find the general solution of the associated homogeneous equation.

   (b) Compute the Wronskian of the solutions you found in part (a) and explain why they form a fundamental set of solutions.

   (c) Compute the general solution of the nonhomogeneous equation. (Use the method of undetermined coefficients.)

   (d) Compute the general solution of the nonhomogeneous equation. (Use the method of variation of parameters.)

   (e) Consider the initial conditions

$$y(0) = 0\,, \qquad y'(0) = 1\,,$$

   and solve the associated initial value problem.

6. Consider the second-order differential equation

$$y'' + 2y' + y = e^{-t} + \sin(t)\,.$$

   (a) Find the general solution of the associated homogeneous equation.

   (b) Compute the Wronskian of the solutions you found in part (a) and explain why they form a fundamental set of solutions.

   (c) Compute the general solution of the nonhomogeneous equation. (Use the method of undetermined coefficients.)

(d) Compute the general solution of the nonhomogeneous equation. (Use the method of variation of parameters.)

(e) Consider the initial conditions

$$y(0) = 1\,, \qquad y'(0) = -1\,,$$

and solve the associated initial value problem.

7. Consider the second-order differential equation

$$y'' + y' + y = t^2\,.$$

(a) Find the general solution of the associated homogeneous equation.

(b) Compute the Wronskian of the solutions you found in part (a) and explain why they form a fundamental set of solutions.

(c) Compute the general solution of the nonhomogeneous equation. (Use the method of undetermined coefficients.)

(d) Compute the general solution of the nonhomogeneous equation. (Use the method of variation of parameters.)

(e) Consider the initial conditions

$$y(0) = 1\,, \qquad y'(0) = 0\,,$$

and solve the associated initial value problem.

4 Series Solutions of Linear Equations (Chapter 5)

Definition 4.1 *A power series is a function of the form*

$$f(t) = \sum_{n=0}^{\infty} a_n \,(t - t_o)^n \ ,$$

where t_o and a_n are real numbers and t is the variable.

Definition 4.2 *The power series is said to converge at $t = c \in \mathbb{R}$ if the series*

$$\sum_{n=0}^{\infty} a_n \,(c - t_o)^n \ ,$$

converges, i.e. if

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n \,(c - t_o)^n < \infty \,.$$

Remark 4.3 *Clearly, the power series converges at $t = t_o$. In order to determine other values of t at which the power series converge, one can use the standard convergence tests (such as, comparison test, ratio test, integral test, and so on).*

Definition 4.4 *The radius of convergence of a power series is a non-negative number (it can possibly be ∞ as well) ρ such that the power series converges for $|t - t_o| < \rho$ and diverges for $|t - t_o| > \rho$.*

Definition 4.5 *Let $f(t)$ be a smooth function. The Taylor series of $f(t)$ is the power series centered at $t = t_o$,*

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t_o)}{n!} \,(t - t_o)^n \ .$$

Definition 4.6 *If a function $f(t)$ has a Taylor series centered at $t = t_0$ with radius of convergence $\rho > 0$, it is said to be analytic at $t = t_o$.*

Remark 4.7 *The idea is to find series solutions to second-order linear (homogeneous) differential equations*

$$R(t)y'' + P(t)y' + Q(t)y = 0 \,.$$

The nonhomogeneous case is analogous.

Observe that here we have a coefficient $R(t)$ in front of the equation, to highlight an important feature regarding the point where we decide to center our power series solution.

Definition 4.8 *We say that a point t_o is an ordinary point if $R(t_o) \neq 0$. Otherwise, t_o is a singular point.*

Remark 4.9 *If t_o is an ordinary point, we can divide our differential equation by $R(t)$ to obtain*

$$y'' + p(t)y' + q(t)y = 0\,,$$

where $p(t) = P(t)/R(t)$ and $q(t) = Q(t)/R(t)$ are analytic at t_o.

Theorem 4.10 *Let t_o be an ordinary point. Then, the general solution to the differential equation*

$$y'' + p(t)y' + q(t)y = 0\,,$$

is the power series

$$y(t) = \sum_{n=0}^{\infty} a_n\,(t - t_o)^n = a_o y_1(t) + a_1 y_2(t)\,,$$

where y_1 and y_2 are two power series analytic at t_o which form a fundamental set of solutions.

Remark 4.11 *Once an ordinary point is fixed, we will consider that the solution of the differential equation is a power series as above and plug in this in the equation. Then, we will try to determine all the coefficients a_n (but for a_o and a_1 which will be parameters) by finding a recurrence relation.*

Example 4.12 *Find a series solution to the following second-order linear homogeneous differential equations:*

1. $y'' + y = 0\,.$
2. $y'' - ty = 0\,,$ *around* $t_o = 0.$
3. $y'' - ty = 0\,,$ *around* $t_o = 1.$
4. $(t^2+1)y'' - 4ty' + 6y = 0\,.$
5. $ty'' + y' + ty = 0\,,$ *around* $t_o = 1.$
6. $(1-t)y'' + y = 0\,.$
7. $y'' + ty' + 2y = 0\,.$
8. $2y'' + (t+1)y' + 3y = 0\,,$ *around* $t_o = 2.$

4.1 Exercises

1. Consider the second-order linear homogeneous differential equations

$$R(t)y'' + P(t)y' + Q(t)y = 0\,.$$

A singular point t_o (ie., $R(t_o) = 0$) is called a regular singular point if

$$\lim_{t \to t_o} (t - t_o) \frac{P(t)}{R(t)}, \qquad \text{and} \qquad \lim_{t \to t_o} (t - t_o)^2 \frac{Q(t)}{R(t)},$$

are finite. Near a regular singular point these differential equations behave very much like Euler equations. Hence, we will assume that a solution of type

$$y(t) = (t - t_o)^r \sum_{n=0}^{\infty} a_n (t - t_o)^n = \sum_{n=0}^{\infty} a_n (t - t_o)^{r+n},$$

exists, where $a_o \neq 0$ and r must be determined by plugging $y(t)$ into the equation and setting the coefficient of $(t - t_o)^r$ to be zero (this is known as the indical equation).

(a) Use the above explanation to find the series solution to the differential equation

$$2t^2 y'' - ty' + (1 + t)y = 0,$$

around $t_o = 0$.

(Remark: If the solutions of the indical equation r_1 and r_2 satisfy $r_2 = r_1 + N$, where N is any positive integer, this method will only give one solution.)

2. The second-order linear homogeneous differential equation

$$t^2 y'' + ty' + \left(t^2 - \nu^2\right) y = 0,$$

is called the Bessel's equation of order ν.

(a) Find the solutions to the indical equation (around $t_o = 0$).

(b) For $\nu = 0$, show that one series solution around $t_o = 0$ is

$$J_0(t) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{2^{2n} (n!)^2},$$

which is known as the Bessel function of the first kind of order zero.

(c) For $\nu = 1$, show that one series solution around $t_o = 0$ is

$$J_1(t) = \frac{t}{2} \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{2^{2n} n!(n+1)!},$$

which is known as the Bessel function of the first kind of order one.

3. The second-order linear homogeneous differential equation

$$t(1 - t)y'' + (\gamma - (1 + \alpha + \beta)t)\, y' - \alpha\beta y = 0,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are constants, is known as the hypergeometric equation.

(a) Find the solutions of the indical equation (around $t_o = 0$).

(b) Show that around $t_o = 0$ one solution is

$${}_2F_1(\alpha, \beta, \gamma; t) = 1 + \frac{\alpha\beta}{\gamma} t + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{t^2}{2!} + \cdots,$$

which is the hypergeometric function.

5 Laplace Transforms (Chapter 6)

Definition 5.1 *Let $f(t)$ be a piecewise continuous function. The* <u>*Laplace transform*</u> *of $f(t)$ is*

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt\,,$$

which is a function in the variable s.

Remark 5.2 *See the Appendix for a table with the Laplace transform of the most common functions.*

Example 5.3 *Compute the Laplace transform of the following functions (and check that the table is correct):*

1. $f(t) = 1\,.$
2. $f(t) = e^{at}\,, \quad a \in \mathbb{R}\,.$
3. $f(t) = t^n\,, \quad n \in \mathbb{N}\,.$
4. $f(t) = \sin(at)\,, \quad a \in \mathbb{R}\,.$
5. $f(t) = \cos(at)\,, \quad a \in \mathbb{R}\,.$

Proposition 5.4 *The Laplace transform is a linear operator. That is,*

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]\,,$$

for every $a, b \in \mathbb{R}$ and piecewise continuous functions $f(t)$ and $g(t)$.

Proposition 5.5 *Assume that the functions $f, f', \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is a piecewise continuous function. Then,*

$$\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - \sum_{i=0}^{n-1} s^{n-1-i} f^{(i)}(0)\,.$$

Remark 5.6 *If we expand the sum on the right-hand side above, we get*

$$\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots s f^{(n-2)}(0) - f^{(n-1)}(0)\,.$$

In particular, we have

$$\begin{aligned} \mathcal{L}[f'] &= s\mathcal{L}[f] - f(0)\,, \\ \mathcal{L}[f''] &= s^2\mathcal{L}[f] - sf(0) - f'(0)\,, \end{aligned}$$

which are going to be very useful since we are dealing mainly with second-order differential equations.

5.1 Solving Initial Value Problems with Laplace Transforms

Remark 5.7 *Given an initial value problem, one can employ the Laplace transform to try to find the solution. The first step of this process is to consider the Laplace transform of the differential equation. This substitutes the differential equation by an algebraic equation, for which it is possible to explicitly solve the Laplace transform of the solution, namely,* $\mathcal{L}[y]$*. Observe that prior to solving for* $\mathcal{L}[y]$ *we need to plug in the initial conditions, hence, if the initial conditions are not given at* $t_o = 0$ *we need to apply a change of variable to the initial value problem.*

Once we have $\mathcal{L}[y]$ *explicitly as a function of* s*, we will apply the inverse Laplace transform to recover* y*. For this step, we may need to consider the decomposition in partial fractions and the linearity of the inverse Laplace transform (hence, this process may require some work).*

Remark 5.8 *One of the theoretical advantages of using the Laplace transform is that there are no a priori restrictions on the type of differential equation. However, it will be essential that we know how to compute the Laplace transform and its inverse.*

For instance, it may be possible if the differential equation is linear with constant coefficients (and not necessarily of second order). It may also be possible to apply this method to some 'simple' linear differential equations with nonconstant coefficients.

Remark 5.9 *Another theoretical advantage of employing the Laplace transform is that it directly gives us the solution to the initial value problem, rather than first obtaining the general solution and then using the initial conditions to determine the constants of integration. However, in practice, this method may require quite a lot of work and, hence, sometimes it may be more efficient to apply previous methods (if available).*

Example 5.10 *Use the Laplace transform to find the solution of the following initial value problems:*

1. $y'' - 10y' + 9y = 5t\,, \quad y(0) = -1\,, \quad y'(0) = 2\,.$

2. $2y'' + 3y' - 2y = te^{-2t}\,, \quad y(0) = 0\,, \quad y'(0) = -2\,.$

3. $y'' - 6y' + 15y = 2\sin(3t)\,, \quad y(0) = -1\,, \quad y'(0) = -4\,.$

4. $y'' + 4y' = \cos(t-3) + 4t\,, \quad y(3) = 0\,, \quad y'(3) = 7\,.$

5. $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0\,, \quad y(0) = 0\,, \quad y'(0) = 1\,, \quad y''(0) = 0\,, \quad y'''(0) = 1\,.$

6. $y^{(4)} - y = 0\,, \quad y(0) = 1\,, \quad y'(0) = 0\,, \quad y''(0) = 1\,, \quad y'''(0) = 0\,.$

7. $y'' + 3ty' - 6y = 2\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$

8. $ty'' - ty' + y = 2\,, \quad y(0) = 2\,, \quad y'(0) = -4\,.$

Remark 5.11 *For the Examples 7 and 8 above, we will need to use that*

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}\mathcal{L}[f(t)]\,,$$

and then solve a first order differential equation. This identity follows (formally) just differentiating with respect to s the Laplace transform of $f(t)$.

5.2 Exercises

1. Using Euler's formula
$$e^{ix} = \cos x + i \sin x\,,$$
show that
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}\,, \qquad \text{and} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2}\,.$$
Use these expressions for $\cos x$ and $\sin x$ to compute their Laplace transforms. (Hint: Treat the imaginary unit i as a constant number.)

2. Consider the second-order linear homogeneous differential equation with constant coefficients
$$y'' - y = 0\,.$$
   (a) Find the fundamental set of solutions.
   (b) Find the solution for the initial conditions
   $$y(0) = 0\,, \qquad y'(0) = 1\,.$$
   (c) Find the solution for the initial conditions
   $$y(0) = 1\,, \qquad y'(0) = 0\,.$$
   (d) Verify that the hyperbolic sine $\sinh x$ satisfies the above differential equation and the initial conditions of part (b). Use the uniqueness of solution for this initial value problem to deduce that
   $$\sinh x = \frac{e^x - e^{-x}}{2}\,.$$
   (e) Verify that the hyperbolic cosine $\cosh x$ satisfies the above differential equation and the initial conditions of part (c). Use the uniqueness of solution for this initial value problem to deduce that
   $$\cosh x = \frac{e^x + e^{-x}}{2}\,.$$
   (f) Use the above expressions of $\cosh x$ and $\sinh x$ to compute their Laplace transforms.

3. Check that the Laplace transforms of the remaining functions of the table in the Appendix are correct.

4. The Heaviside function (also known as unit step function) is the function defined by parts

$$u_c(t) = \begin{cases} 0\,, & t < c\,, \\ 1\,, & t \geq c\,, \end{cases}$$

where $c > 0$ is an arbitrary positive constant.

(a) Compute the Laplace transform of $u_c(t)$.

(b) Show that

$$\mathcal{L}\left[u_c(t) f(t-c)\right] = e^{-cs} \mathcal{L}[f(t)]\,,$$

for every function $f(t)$.

(c) Use the Laplace transform to find the solution to the following initial value problem

$$2y'' + y' + 2y = u_5(t) - u_{20}(t)\,, \qquad y(0) = 0\,, \quad y'(0) = 0\,.$$

5. The Dirac's delta $\delta(t)$ is a generalized function defined by the properties that

$$\delta(t) = 0\,,$$

for every $t \neq 0$, and

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1\,.$$

(Observe that $\delta(t)$ is not a function in the classical sense, it is a generalized function or distribution which can be thought to be zero everywhere but at $t = 0$, where it is infinity. It is very useful to model an impulse.)

(a) Convince yourself that the Dirac's delta $\delta(t)$ is the distributional derivative of the Heaviside function $u_0(t)$.

(b) Show that the Dirac's delta $\delta(t)$ can be seen as the limit when $\tau \to 0^+$ of the functions

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau}\,, & -\tau < t < \tau\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

(c) Compute the Laplace transform of $\delta(t - t_o)$ and show the general property

$$\int_{-\infty}^{\infty} \delta(t - t_o) f(t)\, dt = f(t_o)\,,$$

for every function $f(t)$.

6 Higher-Order Linear Differential Equations (Chapter 4)

Remark 6.1 *In this chapter, we will extend the theory and techniques of Chapter 3 regarding second-order linear differential equations to the higher-order case given by the n-th order linear differential equation*

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)\,.$$

Theorem 6.2 (Existence and Uniqueness) *Let $p_1, ..., p_n$ and g be continuous functions on an open interval $\alpha < t < \beta$ containing $t = t_o$. Then, there exists a unique solution of the initial value problem*

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)\,,$$

with the n initial conditions

$$y(t_o) = y_o, \quad y'(t_o) = y'_o, \quad ... \quad y^{(n-1)}(t_o) = y_o^{(n-1)}\,,$$

where $y_o, y'_o, ..., y_o^{(n-1)}$ are arbitrary prescribed functions.

Theorem 6.3 *Let $p_1, ..., p_n$ and g be continuous functions on an open interval. If $y_1, ..., y_n$ are solutions of the n-th order linear homogeneous differential equation*

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0\,,$$

such that the Wronskian

$$W[y_1, ..., y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0\,,$$

everywhere, then $y_1, ..., y_n$ form a fundamental set of solutions.

Moreover, the general solution to the n-th order linear nonhomogeneous differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)\,,$$

is given in the form

$$y(t) = c_1y_1(t) + \cdots + c_ny_n(t) + y_p(t)\,,$$

where y_p is a particular solution.

Remark 6.4 *From now on we will restrict ourselves to the case where the differential equation has constant coefficients, that is,*

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = g(t)\,.$$

From the above result, we first need to find a fundamental set of solutions to the associated homogeneous differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0\,.$$

A natural type of functions to test is $y(t) = e^{rt}$, where r is a parameter to be determined. This yields the characteristic equation

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0\,.$$

Assume one of these roots (say r) is real with multiplicity k. Then, we have the k solutions

$$e^{rt}, \quad te^{rt}, \quad \ldots \quad t^{k-1}e^{rt}\,.$$

If a root is complex $r = \lambda + i\mu$ with multiplicity k, we will have the $2k$ solutions

$$e^{\lambda t}\cos(\mu t), \quad e^{\lambda t}\sin(\mu t), \quad te^{\lambda t}\cos(\mu t), \quad te^{\lambda t}\sin(\mu t), \quad \ldots \quad t^{k-1}e^{\lambda t}\cos(\mu t), \quad t^{k-1}e^{\lambda t}\sin(\mu t)\,.$$

This way we can construct all the n solutions that form the fundamental set of solutions.

To find the particular solution $y_p(t)$ of the nonhomogeneous differential equation we have the method of undetermined coefficients and the method of variation of parameters.

Example 6.5 *Find the general solutions of the following differential equations:*

1. $y''' - 5y'' - 22y' + 56y = 0\,.$
2. $2y^{(4)} + 11y''' + 18y'' + 4y' - 8y = 0\,.$
3. $y^{(5)} + 12y^{(4)} + 10y''' + 408y'' + 1156y' = 0\,.$
   (Hint: Observe that the characteristic equation can be written as $r(r^2 + 6r + 34)^2$.)
4. $y^{(5)} - 15y^{(4)} + 84y''' - 220y'' + 275y' - 125y = 0\,.$
5. $y^{(4)} + y''' - 7y'' - y' + 6y = 0\,.$
6. $y^{(4)} - y = 0\,.$
7. $y^{(4)} + 2y'' + y = 0\,.$
8. $y^{(6)} - y'' = 0\,.$

Remark 6.6 (Method of Undetermined Coefficients) *As for the case of second-order linear differential equations, if the nonhomogeneous term $g(t)$ is an exponential, polynomial, sine, cosine, sum of these and/or product of these, we can guess the form of a particular solution and plug in this guess into the the differential equation to determine the coefficients.*

Remark 6.7 *Recall that if a portion of our guess is already part of the fundamental set of solutions for the associated homogeneous equation, we need to multiply that portion by t (as many times as needed, so we may end up with powers t^k depending on the multiplicity of that particular root of the characteristic equation).*

Example 6.8 *Find the general solution of the following linear differential equations:*

1. $y''' - 3y'' + 3y' - y = 4e^t$.

2. $y^{(4)} + 2y'' + y = 3\sin(t) - 5\cos(t)$.

3. $y''' - 4y' = t + 3\cos(t) + e^{-2t}$.

4. $y''' - 12y'' + 48y' - 64y = 12 - 32e^{-8t} + 2e^{4t}$.

Remark 6.9 (Method of Variation of Parameters) *Consider the n-th order linear differential equation*

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = g(t)\,,$$

and assume that $y_1, ..., y_n$ form a fundamental set of solutions of the associated homogeneous equation.

The method of variation of parameters consist on determining n functions $u_1, ..., u_n$ such that

$$y(t) = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t)\,,$$

is a particular solution of the nonhomogeneous equation.

Computing $y'(t)$ we find the first condition to be imposed:

$$u_1'(t)y_1(t) + \cdots + u_n'(t)y_n(t) = 0\,.$$

Otherwise, the derivatives of $u_i(t)$, $i = 1, ..., n$, will appear and that would complicate our task of determining $u_i(t)$, $i = 1, ..., n$.

Similarly, computing higher-order derivatives of $y(t)$, we would find the following conditions that we need to impose:

$$u_1'(t)y_1^{(m)}(t) + \cdots + u_n'(t)y_n^{(m)}(t) = 0\,,$$

for every $m = 0, 1, ..., n-2$.

In addition, we also have the condition that $y(t)$ must satisfy the nonhomogeneous equation. Since $y_1, ..., y_n$ are solutions of the homogeneous equation, rearranging we get that this new condition reduces to

$$u_1'(t)y_1^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t) = g(t)\,.$$

Combining everything we have the linear system with n equations and n unknowns (namely, $u_1', ..., u_n'$)

$$\begin{aligned} y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' &= 0\,, \\ y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' &= 0\,, \\ &\vdots \quad \vdots \\ y_1^{(n-1)} u_1' + y_2^{(n-1)} u_2' + \cdots + y_n^{(n-1)} u_n' &= g(t)\,. \end{aligned}$$

From Linear Algebra we know that this system has a unique solution as long as the determinant of the coefficient matrix is nonzero. Observe that this determinant is, precisely, the Wronskian $W[y_1, ..., y_n]$, which we know is nonzero since $y_1, ..., y_n$ form a fundamental set of solutions of the associated homogeneous equation.

If we employ Cramer's rule, the solution to the above system is given by

$$u_1'(t) = \frac{W_1(t)g(t)}{W[y_1, ..., y_n]}\,, \quad ..., \quad u_n'(t) = \frac{W_n(t)g(t)}{W[y_1, ..., y_n]}\,,$$

where $W_m(t)$ is the determinant obtained from the Wronskian $W[y_1, ..., y_n]$ by replacing the m-th column by $(0, ..., 0, 1)^T$.

Finally, integrating we deduce that a particular solution can be given by

$$y_p(t) = y_1(t) \int \frac{W_1(t)g(t)}{W[y_1, ..., y_n]}\, dt + \cdots + y_n(t) \int \frac{W_n(t)g(t)}{W[y_1, ..., y_n]}\, dt\,.$$

Example 6.10 *Find the general solution of the following linear differential equations:*

1. $y''' - y'' - y' + y = e^t\,.$
2. $y''' + y' = \tan(t)\,.$
3. $y''' - y' = t\,.$
4. $y''' - 2y'' - y' + 2y = e^{4t}\,.$
5. $y''' - y'' + y' - y = e^{-t}\sin(t)\,.$

Review Problems

1. Consider the second-order differential equation

$$y'' + y' - 2y = te^t .$$

   (a) Find the general solution of the associated homogeneous equation.

   (b) Compute the Wronskian of the solutions you found in part (a) and explain why they form a fundamental set of solutions.

   (c) Find the series solution of the associated homogeneous equation around $t_o = 1$.

   (d) Compute the general solution of the nonhomogeneous equation. (Use the method of undetermined coefficients.)

   (e) Compute the general solution of the nonhomogeneous equation. (Use the method of variation of parameters.)

   (f) Consider the initial conditions

$$y(0) = 0 , \qquad y'(0) = 1 ,$$

   and solve the associated initial value problem.

   (g) Solve the above initial value problem using the Laplace transform.

2. Consider the second-order differential equation

$$y'' + 2y' + y = e^{-t} + \sin(t) .$$

   (a) Find the general solution of the associated homogeneous equation.

   (b) Compute the Wronskian of the solutions you found in part (a) and explain why they form a fundamental set of solutions.

   (c) Find the series solution of the associated homogeneous equation around $t_o = 1$.

   (d) Compute the general solution of the nonhomogeneous equation. (Use the method of undetermined coefficients.)

   (e) Compute the general solution of the nonhomogeneous equation. (Use the method of variation of parameters.)

   (f) Consider the initial conditions

$$y(0) = 1 , \qquad y'(0) = -1 ,$$

   and solve the associated initial value problem.

   (g) Solve the above initial value problem using the Laplace transform.

3. Consider the second-order differential equation

$$y'' + y' + y = t^2 .$$

(a) Find the general solution of the associated homogeneous equation.

(b) Compute the Wronskian of the solutions you found in part (a) and explain why they form a fundamental set of solutions.

(c) Find the series solution of the associated homogeneous equation around $t_o = 1$.

(d) Compute the general solution of the nonhomogeneous equation. (Use the method of undetermined coefficients.)

(e) Compute the general solution of the nonhomogeneous equation. (Use the method of variation of parameters.)

(f) Consider the initial conditions

$$y(0) = 1\,, \qquad y'(0) = 0\,,$$

and solve the associated initial value problem.

(g) Solve the above initial value problem using the Laplace transform.

4. Consider the differential equation

$$y''' + 3y'' + 3y' + y = te^t\,.$$

(a) Find the general solution of the associated homogeneous equation.

(b) Compute the Wronskian of the solutions you found in part (a) and explain why they form a fundamental set of solutions.

(c) Compute the general solution of the nonhomogeneous equation. (Use the method of undetermined coefficients.)

(d) Compute the general solution of the nonhomogeneous equation. (Use the method of variation of parameters.)

(e) Consider the initial conditions

$$y(0) = 1\,, \qquad y'(0) = 0\,, \qquad y''(0) = 1\,,$$

and solve the associated initial value problem.

(f) Solve the above initial value problem using the Laplace transform.

5. Consider the differential equation

$$y''' + y'' + y' + y = \sin(2t)\,.$$

(a) Find the general solution of the associated homogeneous equation.

(b) Compute the Wronskian of the solutions you found in part (a) and explain why they form a fundamental set of solutions.

(c) Compute the general solution of the nonhomogeneous equation. (Use the method of undetermined coefficients.)

(d) Compute the general solution of the nonhomogeneous equation. (Use the method of variation of parameters.)

(e) Consider the initial conditions

$$y(0) = -1\,, \qquad y'(0) = -1\,, \qquad y''(0) = 1\,,$$

and solve the associated initial value problem.

(f) Solve the above initial value problem using the Laplace transform.

6. Consider the differential equation

$$y^{(4)} + 2y''' + 2y'' + 2y' + y = t^3 + \sin(t)\,.$$

(a) Find the general solution of the associated homogeneous equation.

(b) Compute the Wronskian of the solutions you found in part (a) and explain why they form a fundamental set of solutions.

(c) Compute the general solution of the nonhomogeneous equation. (Use the method of undetermined coefficients.)

(d) Compute the general solution of the nonhomogeneous equation. (Use the method of variation of parameters.)

(e) Consider the initial conditions

$$y(0) = 1\,, \qquad y'(0) = 1\,, \qquad y''(0) = 0\,, \qquad y'''(0) = -1\,,$$

and solve the associated initial value problem.

(f) Solve the above initial value problem using the Laplace transform.

Appendix. Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}[f(t)]$		Formula
$f(t) = 1$	$F(s) = \dfrac{1}{s}$	$s > 0$	A
$f(t) = e^{at}$	$F(s) = \dfrac{1}{(s-a)}$	$s > a$	B
$f(t) = t^n$	$F(s) = \dfrac{n!}{s^{(n+1)}}$	$s > 0$	C
$f(t) = \sin(at)$	$F(s) = \dfrac{a}{s^2 + a^2}$	$s > 0$	D
$f(t) = \cos(at)$	$F(s) = \dfrac{s}{s^2 + a^2}$	$s > 0$	E
$f(t) = \sinh(at)$	$F(s) = \dfrac{a}{s^2 - a^2}$	$s > \lvert a \rvert$	F
$f(t) = \cosh(at)$	$F(s) = \dfrac{s}{s^2 - a^2}$	$s > \lvert a \rvert$	G
$f(t) = t^n e^{at}$	$F(s) = \dfrac{n!}{(s-a)^{(n+1)}}$	$s > a$	H
$f(t) = e^{at}\sin(bt)$	$F(s) = \dfrac{b}{(s-a)^2 + b^2}$	$s > a$	I
$f(t) = e^{at}\cos(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 + b^2}$	$s > a$	J
$f(t) = e^{at}\sinh(bt)$	$F(s) = \dfrac{b}{(s-a)^2 - b^2}$	$s - a > \lvert b \rvert$	K
$f(t) = e^{at}\cosh(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 - b^2}$	$s - a > \lvert b \rvert$	L